

GAUSSIAN FLUCTUATIONS FOR RANDOM MATRICES WITH CORRELATED ENTRIES

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ABSTRACT. For random matrix ensembles with non-gaussian matrix elements that may exhibit some correlations, it is shown that centered traces of polynomials in the matrix converge in distribution to a Gaussian process whose covariance matrix is diagonal in the basis of Chebyshev polynomials. The proof is combinatorial and adapts Wigner's argument showing the convergence of the density of states to the semicircle law.

1. INTRODUCTION

Consider an ensemble $X_n = \frac{1}{\sqrt{n}}(a_n(p, q))_{p,q=1\dots n}$ of random $n \times n$ hermitian matrices with matrix elements $a_n(p, q)$ of mean zero and unit variance. A classical result of Wigner [Wig] states that if $a_n(p, q)$, $1 \leq p \leq q \leq n$, are independent and identically distributed (i.i.d.) and satisfy a moment bound, then

$$(1) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \operatorname{Tr}(X_n^k) = \int_{-2}^2 x^k \frac{1}{2\pi} \sqrt{4 - x^2} dx ,$$

namely the limit exists and is equal to the k^{th} moment of the semicircle law. Here \mathbb{E} denotes average with respect to the distribution of the matrix elements $a_n(p, q)$.

Wigner's result pertains to the *density of states*, the large n limit of the empirical eigenvalue distribution $\nu_n(dx) = \frac{1}{n} \sum_{j=1}^n \delta(x - \lambda_n(j))dx$, where $\lambda_n(1) \leq \lambda_n(2) \leq \dots \leq \lambda_n(n)$ are the eigenvalues of X_n . Convergence of the moments (1) implies that ν_n converges weakly, in expectation, to the semicircle law $\frac{1}{2\pi} \sqrt{4 - x^2} \chi_{\{|x| \leq 2\}}(x)dx$. Later L. Arnold showed that under the same hypotheses the convergence holds almost surely, that is, the limit on the l.h.s. of (1) exists also without the expectation \mathbb{E} and is almost surely equal to the r.h.s. [Arn]. Hence Wigner's result on the limit of empirical distributions is in some sense analogous to the law of large numbers of classical probability theory, even though it is linked to the central limit theorem (CLT) of free probability [Voi]. Natural questions thus arise as to the size and type of the fluctuations about this limit.

It was first shown by Jonsson [Jon], albeit for the case of Wishart matrices, that the fluctuations on a suitable scale are Gaussian. In the case of Wigner matrices with i.i.d. entries, it was shown in [SiS, Joh, KKP, BY] that so-called *centered linear statistics*, random variables

$$(2) \quad n \left(\int f(x) \nu_n(dx) - \mathbb{E} \int f(x) \nu_n(dx) \right) = \operatorname{Tr} f(X_n) - \mathbb{E} \operatorname{Tr} f(X_n) ,$$

with f polynomial or even analytic converge in distribution to a Gaussian random variable. Similar results were recently obtained for random band matrices [AZ].

Our recent work [SS] extends Wigner's result on the density of states to a wide class of random matrix ensembles with correlations among the matrix elements. The aim of the present paper is to apply the techniques developed in [SS] in order to prove a central limit theorem for the fluctuations of centered linear statistics as in (2). As described in the next section, this actually holds for a large class of ensembles considered in [SS], but the values of the covariance calculated in the final step of the analysis are model dependent.

The present work also extends results of Johansson [Joh] on the covariance matrix of the limiting Gaussian process for linear statistics of Wigner matrices with Gaussian entries. Recall that the monic Chebyshev polynomials of the first kind are defined through the trigonometric identity

$$T_m(2 \cos(\theta)) = 2 \cos(m\theta).$$

The main result of [Joh] is that, for a Wigner matrix with Gaussian entries a_n of variance s , the monic re-scaled Chebyshev polynomials

$$(3) \quad T_m(x, s) = s^m T_m\left(\frac{x}{s}\right), \quad m \geq 1,$$

diagonalize the covariance matrix. That is

$$\{\mathrm{Tr}(T_m(X_n, s)) - \mathbb{E} \mathrm{Tr}(T_m(X_n, s))\}_{m \geq 1}$$

converges to a family of independent Gaussians. This result was rederived and extended to a multi matrix case by Cabanal-Duvillard using techniques from stochastic integration [CD]. More recently, Kusalik, Mingo and Speicher showed how to obtain this result by combinatorial means from a well-known genus expansion for Gaussian ensembles [KMS]. The results of [Joh] were extended, again by a combinatorial proof, to non-Gaussian ensembles with independent, but not necessarily identical entries in [AZ]. Here we generalize [Joh] to matrices with correlated entries, but suppose that the variances and 4th moments of the entries are identical. As illustrated in [SS], this situation is of interest, *e.g.*, in applications to models of solid state physics.

Our main results are stated after some required technical preliminaries in the next section. To give a flavor of what is there, we present here a CLT for a generalized real symmetric Wigner ensemble defined as follows. For each n , set $[n] = \{1, \dots, n\}$ and let $\phi_n : [n] \rightarrow [n]$ be a map such that, for some $T \in \mathbb{N}$, $\phi_n^T = \mathrm{id}$, but $\#\{p \in [n] \mid \phi_n^t(p) = p\} = o(n)$ for $t = 1, \dots, T-1$. (This condition is empty for $T = 1$.) We suppose that the matrix entries satisfy for $t = 1, \dots, T-1$

$$a_n(p, q) = a_n(q, p) = a_n(\phi_n^t(p), \phi_n^t(q)) = a_n(\phi_n^t(q), \phi_n^t(p)),$$

but are apart from these conditions independent random variables of mean zero, $\mathbb{E}(a_n(p, q)) = 0$, and with moments of all orders (however, not necessarily Gaussian). Furthermore, we assume that the diagonal matrix elements, $a_n(p, p)$, $p = 1, \dots, n$, all have the same variance denoted by $\mathbb{E}(d^2)$, and that the off diagonal elements $a_n(p, q)$, $1 \leq p < q < n$ all have the same variance $\mathbb{E}(a^2)$ and forth moment $\mathbb{E}(a^4)$. The case $\phi_n = \mathrm{id}$ is the classical Wigner ensemble, while the case $\phi_n(p) = n + 1 - p$ with $\phi_n^2 = \mathrm{id}$ is Poirot's flip matrix model [Poi, BMR]. The

density of states of the generalized Wigner ensemble is a centered semicircle law of width $s = \sqrt{\mathbb{E}(a^2)}$ [SS]. Regarding fluctuations of linear statistics, we have

Theorem 1. *For the generalized real symmetric Wigner ensemble, the random variables $\{\text{Tr}(T_m(X_n, s)) - \mathbb{E} \text{Tr}(T_m(X_n, s))\}_{m \geq 1}$, with $s^2 = \mathbb{E}(a^2)$, converge in distribution to a sequence of independent centered Gaussians $\{Y_m\}_{m \geq 1}$ with variances given by*

$$\mathbb{E}(Y_m^2) = \begin{cases} T \mathbb{E}(d^2) & m = 1, \\ 2T (\mathbb{E}(a^4) - [\mathbb{E}(a^2)]^2) & m = 2, \\ 2mT [\mathbb{E}(a^2)]^m & m \geq 3. \end{cases}$$

Remark: In Section 6, we show that Theorem 1 is a consequence of our main result, Theorem 3 below.

As already pointed out, for Gaussian entries and $T = 1$ this result is proved in [Joh, CD] and can be rederived as indicated in [KMS]. The proof in [KMS] is based on the first two terms of a genus expansion for Gaussian ensembles [MN]. Our proof replaces the genus expansion by a combinatorial argument. Theorem 1 shows that the value of the variances is not as universal as the appearance of the semicircle law and the Chebyshev polynomials, or in other words, first and second order freeness in the sense of [KMS]. This will become even more apparent in the next section when we present our main technical results.

The case of complex matrix entries is a bit more complicated to describe and we refer the reader to Section 6. For the classical Wigner ensemble (that is, $T = 1$) with complex matrix entries, the variances are given by $2(\mathbb{E}(|a|^4) - \mathbb{E}(|a|^2)^2)$ for $m = 2$, and for $m \geq 3$ by

$$(4) \quad m \mathbb{E}(|a|^2)^m + m \sum_{k=1}^{\frac{m}{2}} \frac{A_{m-1,k}}{(m-1)!} \left(\mathbb{E}(a^{2k}) \mathbb{E}(\bar{a}^{2(m-k)}) + \delta_{k \neq \frac{m}{2}} \mathbb{E}(a^{2(m-k)}) \mathbb{E}(\bar{a}^{2k}) \right),$$

where $A_{m,k}$ are the Eulerian numbers (see Section 6 for more details).

As in [BY, AZ], the 4th moment of the off-diagonal entries is involved in the variance of Y_2 given in Theorem 1. In particular, we can write

$$\mathbb{E}(Y_2^2) = 4T \mathbb{E}(a^2) + 2T C_4(a),$$

where $C_4(a) = \mathbb{E}(a^4) - 3[\mathbb{E}(a^2)]^2$ is the fourth cumulant (see Section 2), which vanishes in particular if the off-diagonal entries are Gaussian. Also, in contrast to the density of states, the covariance depends on the distribution of the diagonal entries $a_n(p, p)$, through $\mathbb{E}(Y_1^2)$. Indeed the limiting covariance for $m = 1$ is clear, since

$$\begin{aligned} \mathbb{E}(T_1(X, s)^2) &= \mathbb{E}((\text{Tr } X_n)^2) = \frac{1}{n} \sum_{p,q} \mathbb{E}(a_n(p, p) a_n(q, q)) \\ &= \frac{1}{n} \# \{(p, q) \mid q = \phi_n^t(p) \text{ for some } t = 0, 1, \dots, T-1\} \mathbb{E}(d^2). \end{aligned}$$

Already here it is apparent that the existence of CLT type behavior is more universal than the value of the covariance, which is sensitive to the distribution of the diagonal elements.

It is common to scale the diagonal elements so that $\mathbb{E}(d^2) = 2\mathbb{E}(a^2)$. For $T = 1$ and Gaussian entries this gives the Gaussian Orthogonal Ensemble, which was studied in [Joh]. With this choice,

$$\mathbb{E}(Y_m^2) = 2mT[\mathbb{E}(a^2)]^m + \delta_{m,2}2TC_4(a).$$

If $C_4(a) = 0$, in particular for Gaussian entries, the covariance matrix is a multiple of $\text{diag}(1, 2, 3, \dots)$.

The re-scaled Chebyshev polynomials $T_m(x, s)$ are orthogonal with respect to the probability weight

$$w_s(x) = \frac{1}{\pi} \frac{1}{\sqrt{4s^2 - x^2}}$$

on $[-2s, 2s]$. Specifically, if $T_m(x) = T_m(x, 1)$ as above, then an ortho-normal basis for $L^2([-2, 2], w_1(x)dx)$ is given by $\{1, \frac{1}{\sqrt{2}}T_1(x), \frac{1}{\sqrt{2}}T_2(x), \dots\}$, and Theorem 1 is equivalent to the statement that, for polynomial f ,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \ln \mathbb{E} \left(e^{\text{Tr}(f(X_n)) - \mathbb{E} \text{Tr}(f(X_n))} \right) \\ &= \frac{1}{2} \sum_{m=1}^{\infty} s^{-2m} \mathbb{E}(Y_m^2) \left[\frac{1}{2\pi} \int_{-2}^2 f(sx) T_m(x) \frac{dx}{\sqrt{4-x^2}} \right]^2. \end{aligned}$$

Note that only finitely many terms of the sum are non-zero for polynomial f , as T_m is orthogonal to any polynomial of degree less than m . Plugging in the values of $\mathbb{E}(Y_m^2)$ from Theorem 1, and then carrying out the sum over m allows to show that for polynomial f ,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \ln \mathbb{E} \left(e^{\text{Tr}(f(X_n)) - \mathbb{E} \text{Tr}(f(X_n))} \right) \\ &= T \left(\frac{1}{4\pi^2} \int_{-2}^2 \int_{-2}^2 \left(\frac{f(sx) - f(sy)}{x - y} \right)^2 \frac{(4 - xy)}{\sqrt{4 - x^2} \sqrt{4 - y^2}} dx dy \right. \\ &+ \frac{C_4(a)}{\mathbb{E}(a^2)^2} \left[\frac{1}{2\pi} \int_{-2}^2 f(sx) (x^2 - 2) \frac{dx}{\sqrt{4 - x^2}} \right]^2 \\ &+ \left. \left(\frac{\mathbb{E}(d^2)}{2\mathbb{E}(a^2)} - 1 \right) \left[\frac{1}{2\pi} \int_{-2}^2 f(sx) x \frac{dx}{\sqrt{4 - x^2}} \right]^2 \right). \end{aligned}$$

It is an interesting question, which we do not address here, whether this identity holds for non-polynomial f from some larger class of functions.

2. MAIN TECHNICAL RESULTS

We begin by introducing a class of ensembles of hermitian random matrices $X_n = \frac{1}{\sqrt{n}}(a_n(p, q))_{p,q=1,\dots,n}$, $a_n(p, q) = \overline{a_n(q, p)}$ along the lines of [SS]. For each $n \in \mathbb{N}$, suppose given an equivalence relation \sim_n on pairs $P = (p, q)$ of indices in $[n]^2 = \{1, \dots, n\}^{\times 2}$, satisfying $(p, q) \sim_n (q, p)$. The entries of X_n are supposed to be complex random variables with $a_n(p_1, q_1), \dots, a_n(p_j, q_j)$ independent whenever $(p_1, q_1), \dots, (p_j, q_j)$ belong to j distinct equivalence classes of the relation \sim_n . Furthermore, we assume $a_n(p, q)$ to be centered and to satisfy the moment condition

$$(5) \quad m_k = \sup_n \max_{p,q=1,\dots,n} \mathbb{E}(|a_n(p,q)|^k) < \infty,$$

for all $k \in \mathbb{N}$. We shall also assume that there is a fixed $s > 0$ such that

$$(6) \quad \mathbb{E}(|a_n(p,q)|^2) = s^2, \quad \text{for } p \neq q.$$

For equivalent pairs $(p,q) \sim_n (p',q')$, the relation between $a_n(p,q)$ and $a_n(p',q')$ is not specified; these variables may be identical or correlated.

Properties of the random matrix ensemble depend on combinatorics of the equivalence relations \sim_n . Specifically, we define statistics which count the number of solutions to the equation

$$(7) \quad (p,q) \sim_n (p',q')$$

given one, two or three of the terms:

$$(8) \quad \alpha_1(n) = \max_{p \in [n]} \#\{(q,p',q') \in [n]^3 \mid (7) \text{ holds}\}$$

$$(9) \quad \alpha_2(n) = \max_{p,q \in [n]^2} \#\{(p',q') \in [n]^2 \mid (7) \text{ holds}\}$$

$$(10) \quad \alpha_3(n) = \max_{p,q,p' \in [n]^3} \#\{q' \in [n] \mid (7) \text{ holds}\}.$$

Analogously $\alpha_0(n)$ would count the number of equivalent pairs, which is at least n^2 since $(p,q) \sim_n (q,p)$. A more useful quantity for our analysis is the number of pairs $(p,q) \sim_n (q,p')$ with $p \neq p'$:

$$(11) \quad \hat{\alpha}_0(n) = \#\{(p,q,p') \in [n]^3 \mid (p,q) \sim_n (q,p') \& p \neq p'\}.$$

The main result of [SS] is that the asymptotic density of states of X_n is the centered semi-circle law of width $4s$ provided

$$(12) \quad \alpha_1(n) = o(n^2), \quad \alpha_3(n) = O(1), \quad \text{and} \quad \hat{\alpha}_0(n) = o(n^2).$$

As indicated there, the same result holds — by essentially the same proof — provided

$$(13) \quad \alpha_1(n) = O(n^{2-\epsilon}), \quad \alpha_3(n) = O(n^\epsilon), \quad \text{and} \quad \hat{\alpha}_0(n) = o(n^2)$$

for all sufficiently small ϵ .

For the present work, we impose the following stronger conditions on the equivalence relation:

$$(C1) \quad \alpha_2(n) = O(n^\epsilon)$$

for all $\epsilon > 0$, and

$$(C2) \quad \hat{\alpha}_0(n) = O(n^{2-\delta})$$

for some $\delta > 0$. Note that $\alpha_1(n) \leq n\alpha_2(n)$, so the first condition (C1) is quite a bit more restrictive than the condition on $\alpha_1(n)$ imposed in [SS]. In particular, (13) holds, and by [SS] $\frac{1}{n}\mathbb{E} \text{Tr}((X_n)^k)$ converges to the k th moment of a semicircle law. Here we are interested in the fluctuations around this convergence.

It is a classical fact that a finite or countable set of random variables $\{Y_1, Y_2, \dots\}$ is a joint Gaussian family if and only if all *joint cumulants* among the Y_i of order greater than or equal to three vanish,

$$C_j(Y_{i_1}, \dots, Y_{i_j}) = 0 \quad \text{for all } j \geq 3 \text{ and } i_1, i_2, \dots, i_j = 1, 2, \dots$$

Here the joint cumulant C_j of order $j \geq 1$ is the following multilinear functional defined on \mathbb{C}^j valued random variables (with finite j^{th} moments):

$$C_j(Y_1, \dots, Y_j) = \sum_{\pi \in \mathcal{P}_{[j]}} (-1)^{\#\pi-1} (\#\pi - 1)! \prod_{l=1}^{\#\pi} \mathbb{E} \left(\prod_{i \in B_l} Y_i \right),$$

where $\mathcal{P}_{[n]}$ denotes the set of partitions of $[n] = \{1, \dots, n\}$ and $\#\pi$ the number of blocks of a partition $\pi = \{B_1, \dots, B_{\#\pi}\}$ with blocks B_l , $l = 1, \dots, \#\pi$. In particular, $C_1(X) = \mathbb{E}(X)$ is the mean of X and $C_2(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$ is the covariance of X and Y . A key virtue of the cumulants is the following: if (Y_1, \dots, Y_j) can be split into two non-empty disjoint sets $(Y_{i_1}, \dots, Y_{i_l})$ and $(Y'_{i'_1}, \dots, Y'_{i'_{l'}})$ which are stochastically independent, then the cumulant vanishes. For matrix valued random variables, we use the compactified notation

$$C_j(X_1, \dots, X_j) = C_j(\text{Tr}(X_1), \dots, \text{Tr}(X_j)).$$

By the characterization of Gaussian families through vanishing of higher cumulants, the following theorem shows that any joint limit of $\{\text{Tr}(X_n^k) \mid k \geq 1\}$ is Gaussian (should such a limit exist).

Theorem 2. *For an ensemble of random matrices satisfying condition (C1), one has for $j \geq 3$ and any integer powers $k_1, \dots, k_j \geq 1$,*

$$C_j(X_n^{k_1}, \dots, X_n^{k_j}) = o(1).$$

Remark: Here and below $o(1)$ indicates an error term which vanishes in the limit $n \rightarrow \infty$, but the speed of convergence depends on k_1, \dots, k_j and the asymptotics in (C1) (and below on (C2)).

Theorem 2 already implies a weak version of a central limit theorem, at least with a non-triviality assumption on the covariance matrix $C_2(X_n^{k_1}, X_n^{k_2})$. For any subset $S \subset \mathbb{N}$, let

$$M_n^S(k_1, k_2) = C_2(X_n^{k_1}, X_n^{k_2}), \quad k_1, k_2 \in S.$$

Thus for finite S , M_n^S is a symmetric, positive semi-definite matrix. Furthermore

Corollary 1. *If $\limsup_{n \rightarrow \infty} \| (M_n^S)^{-1} \| < \infty$ for some finite subset $S \subset \mathbb{N}$, then*

$$Y_n(k) = \sum_{k' \in S} (M_n^S)^{-\frac{1}{2}}(k, k')[\text{Tr}(X_n^{k'}) - \mathbb{E}(\text{Tr}(X_n^{k'}))], \quad k \in S,$$

converge in distribution $n \rightarrow \infty$ to a family of independent, centered Gaussians with unit variance.

Proof. It follows from Theorem 2, the multi-linearity of the cumulants, and the bound $(M_n^S)^{-\frac{1}{2}}(k, k') \leq \| (M_n^S)^{-1} \|^\frac{1}{2}$, that

$$C_j(Y_n(k_1), \dots, Y_n(k_j)) = o(1), \quad \text{for any } k_1, \dots, k_j \in S \text{ and } j \geq 3.$$

But, $\mathbb{E}(Y_n(k)) = 0$ and $C_2(Y_n(k), Y_n(k')) = \delta_{k,k'}$ for every n by construction. Thus the joint cumulants of the $Y_n(k)$ converge to the joint cumulants of a Gaussian family, which is sufficient for convergence in distribution. \square

Thus the fluctuations of the family $\text{Tr}(X_n^k)$, $k \in \mathbb{N}$, are controlled by the covariance matrix $C_2(X_n^{k_1}, X_n^{k_2})$, provided we show that the covariance matrix remains non-singular in the limit $n \rightarrow \infty$. Therefore, our aim then is to evaluate the covariance matrix as far as possible. To state our main result in this respect, some notation needs to be introduced and this also allows to show the first step of the proof of Theorem 2. To begin, we write out the traces and matrix multiplications explicitly:

$$C_j(X_n^{k_1}, \dots, X_n^{k_j}) = \frac{1}{n^{\frac{k}{2}}} \sum_{\mathbf{P}}^* C_j \left(\prod_{\ell=1}^{k_1} a_n(P_{1,\ell}), \dots, \prod_{\ell=1}^{k_j} a_n(P_{j,\ell}) \right) ,$$

where $k = k_1 + \dots + k_j$ and Σ^* denotes the sum over multi-indices $\mathbf{P} = \{P_{i,\ell}\}_{i=1, \dots, j}^{\ell=1, \dots, k_i}$ with index pairs $P_{i,\ell} = (p_{i,\ell}, q_{i,\ell}) \in [n]^2$ satisfying the *consistency relations* $q_{i,\ell} = p_{i,\ell+1}$ and $q_{i,k_i} = p_{i,1}$ that stem from the matrix products and the traces. In the sequel, we shall refer to $[k_i]$ as the i^{th} circle, reflecting the cyclic consistency relation of the associated indices.

The crucial step is to classify the consistent indices \mathbf{P} in this sum according to the partition they induce via \sim_n . Let $\mathcal{P}_{[k_1] \cup \dots \cup [k_j]}$ be the set of partitions of the disjoint union of j distinct circles $[k_i]$, $i = 1, \dots, j$, i.e., the set of partitions of $\{(i, \ell) \mid i = 1, \dots, j \text{ and } \ell = 1, \dots, k_i\}$. By definition, a consistent multi-index \mathbf{P} is compatible with the partition $\pi \in \mathcal{P}_{[k_1] \cup \dots \cup [k_j]}$ if and only if

$$(14) \quad P_{i,\ell} \sim_n P_{i',\ell'} \Leftrightarrow (i, \ell) \sim_\pi (i', \ell') ,$$

where the latter means that (i, ℓ) and (i', ℓ') are in the same block of π . We denote the set of π -compatible consistent multi-indices by $S_n(\pi)$. It follows that $C_j(X_n^{k_1}, \dots, X_n^{k_j})$ is equal to

$$(15) \quad \frac{1}{n^{\frac{k}{2}}} \sum_{\pi \in \mathcal{P}_{[k_1] \cup \dots \cup [k_j]}} \sum_{\mathbf{P} \in S_n(\pi)} C_j \left(\prod_{\ell=1}^{k_1} a_n(P_{1,\ell}), \dots, \prod_{\ell=1}^{k_j} a_n(P_{j,\ell}) \right) .$$

Each partition $\pi \in \mathcal{P}_{[k_1] \cup \dots \cup [k_j]}$ induces a projected partition π_R of the “base space” $[j]$ via

$$i \sim_{\pi_R} i' \Leftrightarrow (i, \ell) \sim_\pi (i', \ell') \text{ for some } \ell \in [k_i] \text{ and } \ell' \in [k_{i'}].$$

We call the partition π *connected* if π_R is the trivial partition consisting of one block. If π is *not* connected and $\mathbf{P} \in S_n(\pi)$, then the variables $\{a_n(P_{i,\ell})\}$ can be separated into at least two disjoint and stochastically independent sets (corresponding to i in distinct blocks of the reduced partition π_R) and thus the contribution to (15) vanishes. Therefore, $C_j(X_n^{k_1}, \dots, X_n^{k_j})$ is equal to

$$(16) \quad \frac{1}{n^{\frac{k}{2}}} \sum_{\pi \in \mathcal{P}_{[k_1] \cup \dots \cup [k_j]}^c} \sum_{\mathbf{P} \in S_n(\pi)} C_j \left(\prod_{\ell=1}^{k_1} a_n(P_{1,\ell}), \dots, \prod_{\ell=1}^{k_j} a_n(P_{j,\ell}) \right) ,$$

where $\mathcal{P}_{[k_1] \cup \dots \cup [k_j]}^c$ denotes the set of connected partitions. The proof of Theorem 2 is completed in the next section starting from this formula.

Our main result on the second cumulants states that a relatively small class of partitions contributes to (16) for large n , namely the so-called dihedral partitions. Recall that the dihedral group D_{2m} , for $m \geq 3$, consists of all rotation and reflection symmetries of a regular polygon with m corners. We identify it with the subgroup of the symmetric group S_m consisting of bijections $g : [m] \rightarrow [m]$ sending neighboring points to neighboring points. Each such bijection in turn induces a connected pair partition $\hat{\pi}_g \in \mathcal{P}_{[m] \cup [m]}$ with blocks $\{(1, \ell), (2, g(\ell))\}$ for $\ell = 1, \dots, m$. Hence one can identify the dihedral group D_{2m} with a subset of $\mathcal{P}_{[m] \cup [m]}^c$.

Theorem 3. *Consider an ensemble of random matrices satisfying conditions (C1) and (C2) and suppose that the second moments of the matrix elements satisfy (6). Then the covariance matrix of the re-scaled Chebyshev polynomials $T_m(x, s)$ satisfies for $m, l \geq 1$*

$$C_2(T_m(X_n, s), T_l(X_n, s)) = \delta_{m=l} V_m + o(1),$$

with

$$(17) \quad V_n(m) = \begin{cases} \frac{1}{n} \sum_{p,q} \mathbb{E}(a_n(p, p)a_n(q, q)) & m = 1, \\ \frac{1}{n^2} \sum_{\substack{p,p' \sim_n (q,q) \\ p \neq q, p' \neq q'}} C_2(|a_n(p, q)|^2, |a_n(p', q')|^2) & m = 2, \\ \frac{1}{n^m} \sum_{g \in D_{2m}} \sum_{\mathbf{P} \in S_n^{\text{OD}}(\hat{\pi}_g)} \prod_{k=1}^m \mathbb{E}(a_n(P_{1,k})a_n(P_{2,g(k)})) & m \geq 3, \end{cases}$$

where $S_n^{\text{OD}}(\hat{\pi}_g)$ is the set of off-diagonal $\hat{\pi}_g$ -compatible consistent multi-indices \mathbf{P} , namely $p_{i,k} \neq q_{i,k}$ where $\mathbf{P} = (P_{i,k})_{i=1,2, k=1,\dots,m} = ((p_{i,k}, q_{i,k}))_{i=1,2, k=1,\dots,m}$.

Actually $S_n^{\text{OD}}(\hat{\pi}_g)$ can be replaced by $S_n(\hat{\pi}_g)$ because the contribution from the diagonal terms vanishes in the limit. Along the lines of Corollary 1 we have:

Corollary 2. *If $\liminf_{n \rightarrow \infty} V_n(m) > 0$ for m in some (finite or infinite) subset $S \subset \mathbb{N}$, then*

$$m \in S \mapsto Y_n(m) = \frac{1}{\sqrt{V_n(m)}} [\text{Tr}(T_m(X, s)) - \mathbb{E}(\text{Tr}(T_m(X, s)))]$$

converges in distribution as $n \rightarrow \infty$ to a family of independent, centered Gaussians with unit variance.

For each $m \geq 0$, the Chebyshev polynomial $T_m(x, s)$ is monic of order m ,

$$(18) \quad T_m(x, s) = \sum_{k=0}^m s^{m-k} T_{m,k} x^k, \quad T_{m,m} = 1,$$

with coefficients $T_{m,k}$ for $k \leq m$. Let T denote the lower triangular matrix with ones on the diagonal and entries $T_{m,k}$ below the diagonal,

$$T_{m,k} = \begin{pmatrix} 1 & 0 & \cdots & & \\ T_{1,0} & 1 & 0 & \cdots & \\ T_{2,0} & T_{2,1} & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Since T is lower triangular, it is invertible with a lower triangular inverse because one can compute this inverse by inverting *finite* lower triangular matrices. Hence (17) is equivalent to the statement that

$$C_2(X_n^{k_1}, X_n^{k_2}) = \sum_{m=1}^{\min\{k_1, k_2\}} (T^{-1})_{k_1, m} (T^{-1})_{k_2, m} V_n(m) + o(1).$$

Symbolically, the covariance matrix M_n with entries $C_2(X_n^{k_1}, X_n^{k_2})$ satisfies

$$M_n = T^{-1} V_n [T^{-1}]^* + o(1),$$

where V_n the diagonal matrix with entries $V_n(m)$.

Since arbitrary equivalent pairs of indices appear in the expression for the covariance $V_n(m)$ and we have not specified the relation between correlated matrix elements $a_n(P)$ and $a_n(P')$ corresponding to equivalent pairs $P \sim_n P'$, it is not really possible to further evaluate the covariance, or even to verify that $\liminf V_n(m) > 0$, without additional assumptions.

However, with further restrictions on the distribution of the matrix elements or the combinatorics of the equivalence relations \sim_n it is of course possible to obtain a stronger result. Theorem 1 is one result of this type. The following corollary is intermediate between Theorem 3 and Theorem 1.

Corollary 3. *Suppose the matrix elements $a_n(p, q)$ are real, that the diagonal elements, $a_n(p, p)$, $p = 1, \dots, n$, all have variance $\mathbb{E}(d^2)$, and that the off-diagonal elements, $a_n(p, q)$, $p \neq q \in \{1, \dots, n\}$, all have variance $\mathbb{E}(a^2)$ and fourth moment $\mathbb{E}(a^4)$. If, furthermore, elements with equivalent indices are equal, that is $(p, q) \sim_n (p', q') \Rightarrow a_n(p, q) = a_n(p', q')$, then the covariance matrix of the re-scaled Chebyshev polynomials satisfies*

$$C_2(T_m(X_n, s), T_l(X_n, s)) = \delta_{m=l} V_n(m) + o(1),$$

with

$$V_n(1) = \mathbb{E}(d^2) \frac{\#\{(p, q) \mid (p, p) \sim_n (q, q)\}}{n},$$

$$V_n(2) = [\mathbb{E}(a^4) - \mathbb{E}(a^2)^2] \frac{\#\{(p, q, p', q') \mid p \neq q, p' \neq q', \text{ and } (p, q) \sim_n (p', q')\}}{n^2},$$

and

$$V_n(m) = 2m \mathbb{E}(a^2)^{2m} \frac{\#S_n^{\text{OD}}(\hat{\pi}_g)}{n^m}, \quad m \geq 3,$$

where g is an arbitrary element of D_{2m} . (The expression does not depend on the choice of g .) In particular

$$\liminf_{n \rightarrow \infty} V_n(m) > 0$$

provided $\mathbb{E}(a^2) > 0$ (for $m \geq 3$), $\mathbb{E}(d^2) > 0$ (for $m = 1$), and $\mathbb{E}(a^4) > \mathbb{E}(a^2)^2$ (for $m = 2$).

Remark: $V_n(2)$ vanishes if and only if $\mathbb{E}(a^4) = \mathbb{E}(a^2)^2$, which holds for a non-constant random variable a if and only if it is a Bernoulli variable taking values $\pm\alpha$ with probability $\frac{1}{2}$. As we show in Section 6, Theorem 1 is an easy consequence of Corollary 3.

Proof of Corollary 3. The expressions for $V_n(1)$ and $V_n(2)$ follow easily from (17). The only subtlety for $V_n(m)$, $m \geq 3$, is the fact that $\#S_n^{\text{OD}}(\hat{\pi}_g)$ is independent of $g \in D_{2m}$. To see this, we define an action of D_{2m} on consistent multi-indices. The dihedral group D_{2m} consists of m rotations and m reflections (these are the elements which are, respectively, even or odd as a permutation). Given a consistent multi-index $\mathbf{P} = ((p_{i,\ell}, q_{i,\ell}))_{i=1,2}^{\ell=1,\dots,m}$, we define $g \cdot \mathbf{P} = ((p'_{i,\ell}, q'_{i,\ell}))_{i=1,2}^{\ell=1,\dots,m}$ as follows:

$$(19) \quad \begin{aligned} (p'_{1,\ell}, q'_{1,\ell}) &= (p_{1,\ell}, q_{1,\ell}), \\ (p'_{2,\ell}, q'_{2,\ell}) &= \begin{cases} (p'_{2,g^{-1}(\ell)}, q'_{2,g^{-1}(\ell)}) & \text{if } g \text{ is a rotation,} \\ (q'_{2,g^{-1}(\ell)}, p'_{2,g^{-1}(\ell)}) & \text{if } g \text{ is a reflection.} \end{cases} \end{aligned}$$

Note that for a reflection the order of the indices is reversed on the second circle. One easily checks that, if $g_1, g_2 \in D_{2m}$ and $\mathbf{P} \in S_n^{\text{OD}}(\hat{\pi}_{g_1})$ then $g_2 \cdot \mathbf{P} \in S_n^{\text{OD}}(\hat{\pi}_{g_2 g_1})$. (It is useful to note that $g^{-1}(\ell+1) = \ell \pm 1$, with sign +1 or -1 if g is, respectively, a rotation or reflection.) It follows that $\#S_n^{\text{OD}}(\hat{\pi}_g)$ is independent of $g \in D_{2m}$. \square

This completes the discussion of our main results. Before turning to the proofs, we remark on two extensions that are possible.

Remark: (Multimatrix case) In [CD, KMS] a multimatrix case has been considered. This means that matrices $X_{c,n}$ are drawn from independent ensembles each carrying a colour index c . If the ensembles satisfy (C1) and (C2) and one includes the colour index on the l.h.s. of the definition (14), the cumulants of products $X_{c_1,n} \cdots X_{c_k,n}$ can be controlled in the same way. The variance of mixed terms is then diagonalized by Chebyshev polynomials of the first kind (instead of second kind). We do not give further details, since the explanations in [KMS] are very complete and the involved combinatorics (of non-crossing linear half pair partitions, in the terminology of [KMS]) are simpler than what we have to consider in Section 5.

Remark: (Sparse random matrix) Let $\gamma \in [0, 1]$. One can modify the random matrix ensemble to $\hat{X}_n = n^{\frac{\gamma-1}{2}} (a_n(p, q) b_n(p, q))_{p,q=1,\dots,n}$ where the a_n 's are as above and the b_n 's are apart from the symmetry condition independent Bernoulli variables taking the value 1 with probability $\frac{1}{n^\gamma}$ and 0 with probability $1 - \frac{1}{n^\gamma}$. The arguments of [SS] and the present paper carry over directly, implying, in particular, that the density of states is still a semicircle law and that the matrices are asymptotically free. The matrices in this ensemble are typically sparse if $\gamma > 0$. For the extreme value $\gamma = 1$ there are only of order n non-vanishing matrix elements.

3. COUNTING INDICES — THE PROOF OF THEOREM 2

For a given partition $\pi \in \mathcal{P}_{[k_1] \cup \dots \cup [k_j]}$, let us call a point (i, ℓ) a *connector* of π if $(i, \ell) \sim_\pi (i', \ell')$ for some $i' \neq i$ and some $\ell' \in [k_{i'}]$. A connector is called *simple*

if it is not linked to any other point on the same circle, $(i, \ell) \not\sim_{\pi} (i, \ell')$ for any $\ell \neq \ell'$. The simple connectors play a key role in controlling the combinatorics of π -compatible multi-indices.

Proof of Theorem 2. Let $j \geq 3$. Consider a partition $\pi \in \mathcal{P}_{[k_1] \cup \dots \cup [k_j]}^c$ and the corresponding term of (16). Since π is connected, every circle has at least one connector. Furthermore, if π has a block consisting of a single point (i, ℓ) the contribution vanishes because the corresponding random variable $a_n(P_{i,\ell})$ is centered and independent of all others so the expectation vanishes. Hence we need only consider a partition π with the number of blocks $\#\pi \leq \frac{k}{2}$ where $k = k_1 + \dots + k_j$.

Let us count the number of indices $\mathbf{P} \in S_n(\pi)$. Starting with $P_{1,1}$, there are n^2 possible values for the indices $P_{1,1} = (p_{1,1}, q_{1,1})$ (if $k_1 = 1$ there are only n choices which just improves the argument below). Now proceed cyclically around the first circle $[k_1]$. At $(1, 2)$, the first index of $P_{1,2} = (p_{1,2}, q_{1,2})$ is already fixed, by consistency ($q_{1,1} = p_{1,2}$). The second can take at most n different values unless $(1, 2)$ is in the same block of π as $(1, 1)$, i.e., $(1, 2) \sim_{\pi} (1, 1)$, in which case it is constrained to at most $\alpha_3(n)$ values. Proceed similarly to $(1, 3)$, etc. At any point $(1, \ell)$, there are at most n free index values, unless the block of $(1, \ell)$ was reached before, in which case there are only $\alpha_3(n)$ possible values for the second index of $P_{1,\ell}$. When the first circle is labeled, choose a connector $(1, \ell) \sim_{\pi} (i, \ell')$ to another circle $[i]$. (To obtain an upper bound, we ignore here the consistency condition at the closure of the circle, namely that $q_{k_1} = p_1$ where $P_{1,k_1} = (p_{k_1}, q_{k_1})$.)

Both indices $P_{i,\ell'} = (p_{i,\ell'}, p_{i,\ell'+1})$ at the first point on the new circle can take only $\alpha_2(n)$ possible values. We proceed cyclically around the new circle and count the free indices as above, and then move via a connector, either on circle 1 or 2, to another circle. As the partition is connected, all circles can be reached using this procedure. Since $\alpha_3(n) \leq \alpha_2(n)$, we conclude by (C1) that there are at most

$$(20) \quad S_n(\pi) \leq \underbrace{n^2}_{\text{start}} \times \underbrace{n^{\#\pi-1}}_{\text{new blocks}} \times \underbrace{\alpha_2(n)^{k-\#\pi}}_{\text{old blocks}} = O(n^{\#\pi+1+\epsilon})$$

π -compatible multi-indices for any $\epsilon > 0$. Due to the prefactor $n^{-\frac{k}{2}}$ in (16), this shows that the contribution from π is $o(1)$ unless $\#\pi \geq \frac{k}{2} - 1$, since by the moment bound (5)

$$\left| C_j \left(\prod_{\ell=1}^{k_1} a_n(P_{1,\ell}), \dots, \prod_{\ell=1}^{k_j} a_n(P_{j,\ell}) \right) \right| \leq \text{const. } m_k .$$

Thus, the remaining possibly non-trivial contributions to (16) are from connected partitions with $\#\pi$ equal to $\frac{k}{2}$, $\frac{k-1}{2}$ and $\frac{k}{2} - 1$. Because the contribution vanishes if π has a singleton block, the following holds: if $\#\pi = \frac{k}{2}$, the partition π must be a pair partition (all blocks contain exactly 2 elements); if $\#\pi = \frac{k-1}{2}$ it must have a single block of size 3 and otherwise be a pair partition; and if $\#\pi = \frac{k}{2} - 1$ it must have either one block of size 4 or two blocks of size 3 apart from pairs. In each of these possibilities, π has a simple connector. Indeed, for a pair partition, every connector is simple. If π has only blocks of size 2 and 3, a simple connector exists since for any given connector $(i, \ell) \sim_{\pi} (i', \ell')$ with $i \neq i'$ either (i, ℓ) or (i', ℓ') is simple. Finally, if π has a single block of size 4 but is otherwise a pair partition,

then a simple connector exists for $j \geq 3$ since either there is a 2-block connecting distinct circles or the 4-block connects all circles. In the later case, we must have $j = 3$ or 4 and at least two of the points in the 4-block are simple connectors. (For $j = 2$ it can happen that there is no simple connector, a fact that will play a key role in the evaluation of the covariance.)

Now suppose that π has a simple connector, which after suitable relabeling we take to be $(1, k_1)$. If $k_1 = 1$, then there are only n choices for $P_{1,1}$. Otherwise start counting the indices as above but stop at $(1, k_1 - 1)$. As $(1, k_1)$ is a simple connector, its block was not yet reached. However, P_{1,k_1} is fixed by consistency, since both of its neighbors $P_{1,1}$ and P_{1,k_1-1} are specified. Consequently the block of $(1, k_1)$ does not contribute a factor n , so we have at most $S_n(\pi) = O(n^{\#\pi+\epsilon})$ consistent π -compatible indices, that is, one power better than in (20). Thus the cases $\#\pi = \frac{k-1}{2}$ and $\frac{k}{2} - 1$ give negligible contributions.

In the case $\#\pi = \frac{k}{2}$, with π is a connected pair partition, the above argument is not quite sufficient. However, because $j \geq 3$ there are at least *two* simple connectors, allowing to reduce the power of n yet again. Indeed, as π is connected there is a circle connected to two distinct circles by simple connectors. After suitable relabeling, suppose this is circle 2 and that it is connected to circles 1 and 3 via connectors $(2, 1) \sim_\pi (1, k_1)$ and $(2, \ell) \sim (3, 1)$. We choose indices as above starting with $P_{1,1}$ so that P_{1,k_1} is specified by consistency and the 2-block $(1, k_1) \sim_\pi (2, 1)$ has no free index. Then $P_{2,1}$ is constrained to $\alpha_2(n)$ possible values. Now choose indices $P_{2,2}, P_{2,3}, \dots, P_{2,\ell-1}$ and similarly for $P_{2,k_2}, P_{k_2-1}, \dots, P_{2,\ell+1}$. As with P_{1,k_1} , both indices in $P_{2,\ell}$ are specified by consistency so the 2-block $(2, \ell) \sim_\pi (3, 1)$ also has no free index. Thus we may reduce the power of n in (20) by 2, that is there are $S_n(\pi) = O(n^{\frac{k}{2}-1+\epsilon})$ consistent π -compatible indices. Accounting for the pre-factor $n^{-\frac{k}{2}}$, we see that this contribution is also $o(1)$. \square

4. PAIR PARTITIONS AND THE COVARIANCE

We now focus on the covariance, that is $j = 2$. A number of the arguments from the proof of Theorem 2 carry over to this case to show that the contributions to (16) from many partitions are negligible. In particular, the contribution from π is $o(1)$ if (i) π has a singleton, (ii) $\#\pi < \frac{k}{2} - 1$, or (iii) $\#\pi < \frac{k}{2}$ and π has a simple connector. Only two classes of partitions remain:

- (I) connected pair partitions ($\#\pi = \frac{k}{2}$), and
- (II) connected partitions with exactly 4 connectors, comprising a 4-block, and with all other blocks being pairs (2-blocks) of elements from the same circle ($\#\pi = \frac{k}{2} - 1$).

In the latter case, the 4-block necessarily consists of 2 connectors on each circle, as otherwise the partition would have a simple connector.

Let us denote the set of all partitions in these two classes by $\mathcal{PP}_{[k_1] \cup [k_2]}^c$. Thus, up to $o(1)$ errors, the sum over partitions in (16) may be restricted to $\mathcal{PP}_{[k_1] \cup [k_2]}^c$. In particular one sees that the covariance of an even and odd power of X_n vanishes

in the limit

$$C_2(X_n^{k_1}, X_n^{k_2}) = o(1), \quad k_1 = 2\ell_1 \text{ and } k_2 = 2\ell_2 + 1$$

since in this case $k = 2\ell_1 + 2\ell_2 + 1$ is *odd* and the class $\mathcal{PP}_{[k_1] \cup [k_2]}^c$ is empty. (The symbol \mathcal{PP} stands for “pair partition,” which is a slight abuse of notation. However, we shall see in Corollary 4, that a true pair partition with exactly 4 connectors gives $o(1)$ contribution while the contribution from the partition in which these 4 connectors form a 4-block does not vanish! This is related to the appearance of the fourth moment in Theorem 1.)

In this section, we prove a series of lemmas showing that various additional classes of partitions give negligible contribution to (16) as $n \rightarrow \infty$. In the end we will have reduced considerations to the so-called *dihedral non-crossing pair partitions*. These are the starting point for the evaluation of the limiting covariance and the proof of Theorem 3 in the next section.

We may draw a planar diagram representing a partition, in which $[k_1]$ and $[k_2]$ are points on the inner and outer boundaries of an annulus and the connections of π are marked by curves in the annulus. For any connected pair $(i, \ell_1) \sim_\pi (i, \ell_2)$ there are two possible ways for the curve marking this connection to wind around the hole in the center of the annulus. We say that this pair is *crossed*, and that the partition is *crossing*, if no matter how we draw this curve it is intersected by another curve marking a different block of the partition. Our first task is to show that the contribution from crossing pair partitions is negligible.

To give a technical definition of crossing the following notation is useful. Given distinct points $\ell \neq \ell'$ in $[k]$ let $]\ell, \ell'[_k$ denote the *open interval between ℓ and ℓ' in the circle $[k]$* ,

$$(21) \quad]\ell, \ell'[_k = \begin{cases} \emptyset & \ell' = \ell + 1, \\ \{\ell + 1, \dots, \ell' - 1\} & \ell' \neq \ell + 1. \end{cases}$$

In this definition addition is modulo k , consistent with the cyclic nature of indices stemming from the matrix trace. For instance $]\ell, \ell'[_k = \{3, 4, \dots, k\}$. Note that for any $\ell \neq \ell'$

$$]\ell, \ell'[_k \cap]\ell', \ell'[_k = \emptyset, \quad \text{and} \quad]\ell, \ell'[_k \cup]\ell', \ell'[_k = [k] \setminus \{\ell, \ell'\}.$$

It is convenient to introduce the closed and half open intervals as well

$$(22) \quad [\ell, \ell']_k =]\ell, \ell'[_k \cup \{\ell\} \cup \{\ell'\}, \quad [\ell, \ell'[_k =]\ell, \ell'[_k \cup \{\ell\}, \quad \text{and} \quad]\ell, \ell']_k =]\ell, \ell'[_k \cup \{\ell'\}.$$

A partition $\pi \in \mathcal{PP}_{[k_1] \cup [k_2]}^c$ is called *crossing* if there are connected points $(i, \ell_1) \sim_\pi (i, \ell_2)$ on the same circle such that the two intervals $\{i\} \times]\ell_1, \ell_2[_{k_1}$ and $\{i\} \times]\ell_2, \ell_1[_{k_1}$ are connected via π either to each other or to the opposite circle. In other words, there are points $m_1 \in]\ell_1, \ell_2[_{k_1}$ and $m_2 \in]\ell_2, \ell_1[_{k_1}$ such that either

- (I) $(i, m_1) \sim_\pi (i, m_2)$, or
- (II) both (i, m_1) and (i, m_2) are connectors.

(It may happen that (i, ℓ_1) and (i, ℓ_2) are part of a 4-block, the other two points of which lie on the opposite circle, in which case there are no other connectors and π is crossing if and only if (I) holds.)

Lemma 1. *Suppose that (C1) holds and let $\pi \in \mathcal{PP}_{[k_1] \cup [k_2]}^c$ be a crossing pair partition. Then π is subdominant, namely*

$$(23) \quad \frac{1}{n^{\frac{k}{2}}} \#S_n(\pi) = o(1).$$

Proof. We claim that, possibly after relabeling indices, we may assume without loss that $(1, 1) \sim_\pi (1, \ell)$, that there are $m_1 \in]1, \ell[_{k_1}$ and $m_2 \in]\ell, 1[_{k_1}$ such that (I) or (II) above hold, and furthermore that

(A) $(1, m_1)$ is not connected to any other point of $\{1\} \times]1, \ell[_{k_1}$,

and, either

(B₁) π contains a 4-block,

or

(B₂) there is a simple connector $(1, m_3)$ with $m_3 \in]\ell, 1[_{k_1}$.

Indeed, we may relabel indices so that the given crossed pair is $(1, 1) \sim_\pi (1, \ell)$ for some ℓ and then find $m_1 \in]1, \ell[_{k_1}$ and $m_2 \in]\ell, 1[_{k_1}$ such that either case (I) or (II) holds. We will see that (A) and either (B₁) or (B₂) hold, possibly after cyclically permuting the labels of the first circle: $(1, j) \mapsto (1, j - \ell + 1)$ so that $(1, \ell) \mapsto (1, 1)$ and $(1, 1) \mapsto (1, k_1 + 1 - \ell)$.

First, we show that (A) holds by contradiction. Suppose (A) fails. Then there is a 4-block made up of $(1, m_1) \sim_\pi (1, j)$ with $j \in]1, \ell[_{k_1}$ and two points on the opposite circle. Since $\pi \in \mathbf{P}_{[k_1] \cup [k_2]}^c$ these are the only connectors of π . But this contradicts the choice of m_1 and m_2 , since $(1, m_2)$ is not a connector, so case (II) does not hold, and also $(1, m_1) \not\sim_\pi (1, m_2)$, so case (I) does not hold.

Second, we show that if (B₁) fails then (B₂) holds. Thus suppose π has no 4-block — so it is a true pair partition. If case (2) in the definition of crossing holds, then we already have (B₂), with $m_3 = m_2$. If instead case (1) holds, so $(1, m_1) \sim_\pi (1, m_2)$. Then, since π is connected, there is a connector $(1, m_3)$. Furthermore, $m_3 \neq 1$ or ℓ as there is no 4-block. If $m_3 \in]\ell, 1[_{k_1}$, we already have (B₂). Otherwise $m_3 \in]1, \ell[_{k_1}$ and we find that (B₂) holds after we cyclically permute the indices $(1, j) \mapsto (1, j - \ell + 1)$.

Thus let us assume we have a partition π with the above properties and count the number of π -compatible consistent multi-indices starting at $(1, 1)$. As in the proof of Theorem 2, there are n^2 choices for $P_{1,1}$ after which we choose in sequence $P_{1,2}, \dots, P_{1,m_1-1}$, gaining each time either a factor n , if we visit a new block, or a factor $\alpha_3(n) \leq \alpha_2(n)$, if we visit an old block. We leave P_{1,m_1} unspecified as yet. Instead we pick $P_{1,\ell}, P_{1,\ell-1}$ on down to P_{1,m_1+1} . There are $\alpha_2(n)$ choices for $P_{1,\ell}$ since $(1, \ell) \sim_\pi (1, 1)$, followed by a factor of n for each new block and a factor of $\alpha_3(n) \leq \alpha_2(n)$ for each old. Since we have chosen P_{1,m_1-1} and P_{1,m_1+1} , both indices $P_{1,m_1} = (p_{1,m_1}, q_{1,m_1})$ are specified by consistency. By (A), the block of $(1, m_1)$ was not previously visited, so this block doesn't contribute a factor of n , allowing us to reduce the power of n on the r.h.s. of (20) by one:

$$S_n(\pi) \leq n^{\#\pi} \alpha_2(n)^{k-\#\pi} = n^{\#\pi+\epsilon}.$$

If π contains a 4-block, that is if (B₁) holds, this already shows that it is subdominant by the arguments at the end of the proof of Theorem 2.

If (B₂) holds, so π contains no 4-block, then the block of the simple connector (1, m_3) also fails to contribute a free index since we may specify P_{1,m_3} by consistency by first choosing $P_{\ell+1}, \dots, P_{1,m_3-1}$ and then $P_{1,k_1}, \dots, P_{1,m_3+1}$. Thus

$$S_n(\pi) \leq n^{\#\pi-1} \alpha_2(n)^{k-\#\pi} = n^{\frac{k}{2}-1+\epsilon},$$

so π is subdominant. \square

A connected partition $\pi \in \mathcal{PP}_{[k_1] \cup [k_2]}^c$ with no crossing is called *non-crossing*. The next lemma, which is essentially the same as Lemma 1 of [SS], is the key to counting $S_n(\pi)$ for a non-crossing partition and is the first place that we apply condition (C2).

Lemma 2. *Assume that (C1) and (C2) hold and suppose $\pi \in \mathcal{PP}_{[k_1] \cup [k_2]}^c$ contains a pair of neighbors which are not part of a four block. That is $(i, \ell) \sim_\pi (i, \ell+1)$ for $i \in \{1, 2\}$ and (i, ℓ) is not connected to any point on the opposite circle. Let $\pi' \in \mathcal{PP}_{[k_1-2] \cup [k_2]}^c$ be the partition obtained by removing the corresponding pair (and relabeling $(i, m) \mapsto (i, m-2)$ for $\ell+2 \leq m \leq k_1$). Then*

$$(24) \quad \frac{1}{n^{\frac{k}{2}}} \#S_n(\pi) \leq \frac{1}{n^{\frac{k}{2}-1}} \#S_n(\pi') + o(1).$$

Remark: One can easily check that the reduced partition π' is non-crossing if and only if π is non-crossing.

Proof. After suitable relabeling, we may assume without loss that the nearest neighbor pair is $(1, k_1-1) \sim_\pi (1, k_1)$. Let us look at the situation close to these points and drop the circle label 1 on the indices involved. The indices are (p_{k_1-2}, p_{k_1-1}) , (p_{k_1-1}, p_{k_1}) , (p_{k_1}, p_1) and (p_1, p_2) . Now consider separately the two cases (i) $p_{k_1-1} = q_1$ and (ii) $p_{k_1-1} \neq p_1$.

In case (i), after eliminating P_{1,k_1-1} and P_{1,k_1} , we have a consistent multi-index which is clearly π' -compatible. Therefore, in this case there are n choices for p_{k_1} followed by at most $\#S_n(\pi')$ choices for the remaining indices, giving the first term in (24). (To obtain an upper bound, we neglect here the condition that the removed pair is not linked to any point in $[k_1-2] \cup [k_2]$.)

There are only $\widehat{\alpha}_0(n)$ triples p_{k_1-1}, p_{k_1}, p_1 which result in case (ii). Suppose we are given such a triple, and let us count the choices for the remaining indices considering separately π that have or don't have a simple connector.

If there is a simple connector, say at $(1, \ell)$, start counting free indices at $(1, 1)$ and proceed to $(1, \ell-1)$. Then start again at $(1, k_1-2)$ and proceed downward to $(1, \ell+1)$ as in the proof of Lemma 1. Then $P_{1,\ell}$ is fixed by consistency, so that this block does not result in a free index. Thus by (C1) and (C2), the number of π -compatible multi-indices falling into case (ii) is bounded by

$$(25) \quad \widehat{\alpha}_0(n)n^{\#\pi-2} \alpha_2(n)^{k-\#\pi} = \widehat{\alpha}_0(n)n^{\frac{k}{2}-2} \alpha_2(n)^{\frac{k}{2}} = O(n^{\frac{k}{2}-\epsilon})$$

for sufficiently small ϵ , giving the second term in (24).

If there is no simple connector (*i.e.*, if π contains a 4-block), we simply start counting free indices at $(1, 1)$ and proceed as in the first part of the proof of Theorem 2 to show that the number of π -compatible indices falling into case (ii) is bounded by

$$(26) \quad \widehat{\alpha}_0(n)n^{\#\pi-1}\alpha_2(n)^{k-\#\pi} = \widehat{\alpha}_0(n)n^{\frac{k}{2}-2}\alpha_2(n)^{\frac{k}{2}+1} = O(n^{\frac{k}{2}-\epsilon}),$$

which is again negligible. \square

We shall apply the nearest neighbor pair reduction of Lemma 2 repeatedly below. Hence, we assume from now on that both (C1) and (C2) hold. The following corollary is ultimately responsible for the appearance of the 4th moment of a in Theorem 1.

Corollary 4. *Let $\pi \in \mathcal{PP}_{[k_1] \cup [k_2]}$ be non-crossing and have only two connectors on each circle. Unless the four connectors form a 4-block, π is subdominant, namely (23) holds.*

Proof. Let $\pi \in \mathcal{PP}_{[k_1] \cup [k_2]}$ be non-crossing with no 4-block but only two connectors on each circle. First we apply Lemma 2 as many times as possible, eliminating all nearest neighbor pairs of π and its resulting descendants. In the end we obtain a non-crossing partition $\pi' \in \mathcal{PP}_{[k'_1] \cup [k'_2]}$ without nearest neighbor pairs such that

$$n^{-\frac{k}{2}}\#S_n(\pi) = n^{-\frac{k'}{2}}\#S_n(\pi') + o(1),$$

where $k = k_1 + k_2$ and $k' = k'_1 + k'_2$. Since π' is non-crossing with two connectors on each circle and has no nearest neighbor pairs, all points must be connectors. Thus $k'_1 = k'_2 = 2$. Furthermore π' has no 4-block (since π has no 4-block). Thus $(1, 1) \not\sim_{\pi'} (1, 2)$ and there are no π' -compatible consistent multi-indices, *i.e.*, $\#S_n(\pi') = 0$. Indeed consistency implies that $P_{1,1} = (p, q)$ and $P_{1,2} = (q, p)$, so $P_{1,1} \sim_n P_{1,2}$ and the indices are not π' -compatible. \square

On the other hand, a partition $\pi' \in \mathcal{PP}_{[2] \cup [2]}^c$ with a 4-block has only one block

$$\{(1, 1), (1, 2), (2, 1), (2, 2)\}.$$

There are (exactly) n^2 choices for $P_{1,1} = (p, q)$, $P_{1,2} = (q, p)$. Once these are specified, we can always take $P_{2,1} = (p, q)$, $P_{2,2} = (q, p)$ or $P_{2,1} = (q, p)$, $P_{2,2} = (p, q)$ to obtain an element of $S_n(\pi')$. Thus

$$(27) \quad n^{-\frac{k'}{2}}\#S_n(\pi') \geq 2,$$

so that π' is indeed dominant.

Motivated by the Corollary 4 and (27), we define

$$\mathcal{NPP}_{[k_1] \cup [k_2]}^2 = \text{non-crossing partitions in } \mathcal{PP}_{[k_1] \cup [k_2]} \text{ with a 4-block.}$$

and for $m \neq 2$,

$$\begin{aligned} \mathcal{NPP}_{[k_1] \cup [k_2]}^m &= \text{non-crossing partitions in } \mathcal{PP}_{[k_1] \cup [k_2]} \\ &\quad \text{with } m \text{ simple connectors on each circle,} \end{aligned}$$

and note that all other partitions are subdominant.

There remains in each $\mathcal{NPP}_{[k_1] \cup [k_2]}^m$ another set of sub-dominant partitions. Essentially these are the partitions with crossings among the connections between the two circles. As above, in the planar diagram representing a partition, the presence of an intersection among the connecting links may depend on the choice of orientation for the points of $[k_1]$ and $[k_2]$ marked on the two circular boundaries of the annular region. If there are no crossings, or if it is possible to redraw the diagram without crossings by reversing one of the circles, we will say the partition is *dihedral*. The terminology here comes from the one to one correspondence explained in Section 2 between the “non-crossing” partitions in $\mathcal{NPP}_{[m] \cup [m]}^m$ and the dihedral group D_{2m} of symmetries of an m -gon.

Dihedral partitions are distinguished by the fact that they connect *neighboring* connectors on one circle with neighboring connectors on the other circle. We call connectors (i, ℓ) and (i, ℓ') on a given circle *neighboring* if there is no other connector in between them, that is if one of $\{i\} \times [\ell, \ell' |_{k_i}]$ or $\{i\} \times [\ell', \ell |_{k_i}]$ contains no connector. We denote by $\mathcal{DNP}_{[k_1] \cup [k_2]}^m$ the set of dihedral partitions of $[k_1] \cup [k_2]$ with m -connectors on each circle.

The following lemma shows that only the dihedral partitions $\mathcal{DNP}_{[k_1] \cup [k_2]}^m$ can potentially contribute in the limit $n \rightarrow \infty$.

Lemma 3. *Suppose that $\pi \in \mathcal{NPP}_{[k_1] \cup [k_2]}^m$ is not dihedral. Then π is subdominant, namely (23) holds.*

Proof. As π is non-crossing, one can apply Lemma 2 to eliminate all pairs and obtain a partition $\pi' \in \mathcal{NPP}_{[m] \cup [m]}^m$ such that all points are connectors. By (24) it suffices to show that π' is subdominant. Furthermore, π' is dihedral if and only if π is dihedral, so we may assume π' is not dihedral. If $m = 1, 2, 3$, all partitions are dihedral and there is nothing to show.

For $m > 3$, after suitable relabeling we may assume that $(1, 1) \sim_{\pi'} (2, 1)$ and $(1, 2) \sim_{\pi'} (2, \ell)$ with $\ell \neq m$ or 2. Start counting indices at $(1, 3)$. There are n^2 choices. Next consider $(1, 4), \dots, (1, m)$. Since every point is a connector, each point is in a new block (as $m > 2$ there is no 4-block). Thus up to now there were no more than $n^2 \times n^{m-3} = n^{m-1}$ choices. Now choose the indices on circle 2 at every point that belongs to a block already accounted for, that is at all points *except* $(2, 1)$ and $(2, \ell)$. For each of these choices there are no more than $\alpha_2(n)$ possibilities. Since we have chosen $P_{2,m}, P_{2,2}, P_{2,\ell-1}$, and $P_{2,\ell}$, the indices at $(2, 1)$ and $(2, \ell)$ are fixed by consistency. Finally, there are only $\alpha_2(n)$ choices for each of $P_{1,1}$ and $P_{1,2}$. In total, we obtain $\#S_n(\pi') \leq n^{m-1} \alpha_2(n)^m$ implying that π' is sub-dominant by condition (C1) and Lemma 2. \square

Let us summarize the results obtained so far by plugging them into (16):

$$(28) \quad C_2(X_n^{k_1}, X_n^{k_2}) \\ = \frac{1}{n^{\frac{k}{2}}} \sum_{m=1}^{\min\{k_1, k_2\}} \sum_{\pi \in \mathcal{DNP}_{[k_1] \cup [k_2]}^m} \sum_{\mathbf{P} \in S_n(\pi)} C_2(a_n(\mathbf{P}_1), a_n(\mathbf{P}_2)) + o(1),$$

where for $i = 1$ or 2

$$(29) \quad a_n(\mathbf{P}_i) = \prod_{\ell=1}^{k_1} a_n(P_{i,\ell}).$$

When $m \neq 2$, each circle has a simple connector, say (i, ℓ_i) . Therefore given $\mathbf{P} \in S_n(\pi)$, the random variables $a_n(P_{i,\ell_i})$, $i = 1, 2$, are paired only with each other so $\mathbb{E}(a_n(\mathbf{P}_i)) = 0$ and we have

$$(30) \quad C_2(a_n(\mathbf{P}_1), a_n(\mathbf{P}_2)) = \mathbb{E}(a_n(\mathbf{P}_1)a_n(\mathbf{P}_2)).$$

However, for $m = 2$ that we must retain the full expression for the covariance

$$(31) \quad C_2(a_n(\mathbf{P}_1), a_n(\mathbf{P}_2)) = \mathbb{E}(a_n(\mathbf{P}_1)a_n(\mathbf{P}_2)) - \mathbb{E}(a_n(\mathbf{P}_1))\mathbb{E}(a_n(\mathbf{P}_2)).$$

In order to evaluate $C_2(a_n(\mathbf{P}_1), a_n(\mathbf{P}_2))$, we need to analyze which kind of π -consistent indices actually contribute. Let $PS_n(\pi)$ denote the set of π -consistent indices \mathbf{P} with the following property:

(P) *For any pair of points on the same circle, $(i, \ell) \sim_\pi (i, \ell')$, we have $P_{i,\ell} = (p, q)$ and $P_{i,\ell'} = (q, p)$ with $p \neq q$.*

Given $\mathbf{P} \in PS_n(\pi)$ and a nearest neighbor pair, say $(i, \ell) \sim_\pi (i, \ell + 1)$, the reduced multi-index \mathbf{P}'

$$(32) \quad P'_{j,k} = \begin{cases} P_{j,k} & j \neq i \\ P_{i,k} & j = i \text{ and } k < \ell \\ P_{i,k+2} & j = i \text{ and } k \geq \ell \end{cases}$$

is in $PS_n(\pi')$ where, as in Lemma 2, π' is the partition obtained by removing the pair $(i, \ell) \sim_\pi (i, \ell + 1)$. Indeed, it is clear that \mathbf{P}' satisfies property (P). The only question is if \mathbf{P}' is consistent. However, this is guaranteed by property (P) for \mathbf{P} , since if $P_{i,\ell} = (p, q)$ and $P_{i,\ell+1} = (q, p)$ then by consistency (of \mathbf{P})

$$P'_{i,\ell-1} = P_{i,\ell-1} = (\cdot, p) \quad \text{and} \quad P'_{i,\ell} = P_{i,\ell+1} = (p, \cdot).$$

We single out the class $PS_n(\pi)$ because the complementary class $NS_n(\pi) = S_n(\pi) \setminus PS_n(\pi)$ gives negligible contribution to (28):

Lemma 4. *Let $\pi \in \mathcal{DNP}\mathcal{P}^m_{[k_1] \cup [k_2]}$. Then*

$$(33) \quad \frac{1}{n^{\frac{k}{2}}} \#NS_n(\pi) = o(1).$$

Remark: Note that $NS_n(\pi)$ includes all multi-indices with a diagonal index (p, p) somewhere. Diagonal indices play a special role because they determine the variance of $\text{Tr}(X)$. However, as far as Lemma 4 is concerned, there is nothing special about diagonal indices. Indeed the result holds, by the same proof, if we pick a subset $B_n \subset [n]^2$ of size $\#B_n = O(n^{2-\delta})$ for some $\delta > 0$ and exclude indices with $(p, q) \in B_n$ from the “good” multi-indices $PS_n(\pi)$.

Proof. We use essentially the same arguments as in the proof of Lemma 2 to eliminate pairs from π , with an additional step since we must consider diagonal matrix

elements separately. As we shall show, this elimination gives

$$(34) \quad \frac{1}{n^{\frac{k}{2}}} \#NS_n(\pi) \leq \frac{1}{n^{\frac{k}{2}}} \#NS_n(\pi') + o(1).$$

After repeated eliminations, we obtain $\pi' \in \mathcal{DNP}P_{[m] \cup [m]}^m$, with $n^{-\frac{k}{2}} \#NS_n(\pi) \leq n^{-m} \#NS_n(\pi') + o(1)$. But all multi-indices $\hat{\mathbf{P}} \in S_n(\pi')$ satisfy (P) — it is an empty condition since all points are connectors under π' . Thus $S_n(\pi') = PS_n(\pi')$ so $NS_n(\pi') = \emptyset$ and (33) holds.

It remains to show (34). If $k_1 = k_2 = m$ then $NS_n(\pi) = \emptyset$ and there is nothing to prove. If $k_1 > m$ or $k_2 > m$, then since π is non-crossing it has a nearest neighbor pair, $(i, \ell) \sim_\pi (i, \ell + 1)$. The corresponding indices in \mathbf{P} are $P_{i,\ell} = (p, q)$ and $P_{i,\ell+1} = (q, p')$. Let us consider three cases: (i) $p = p' \neq q$, (ii) $p \neq p'$, and (iii) $p = p' = q$. (The first two case are cases (i) and (ii) in the proof of Lemma 2.)

In case (i) the indices $P_{i,\ell}$ and $P_{i,\ell+1}$ are “good” — these indices are not the “defect” which prevents \mathbf{P} from being in $PS_n(\pi)$. We conclude that a defect is still present in the reduced multi-index, that is $\mathbf{P}' \in NS_n(\pi')$. Taking q to be a free index, we see that there are no more than $n \times NS_n(\pi'')$ multi-indices in case (i), giving the first term on the r.h.s. of (34).

Finally, we show that there are only $o(1)$ multi-indices which fall in cases (ii) and (iii). Indeed, for case (ii) this was already shown in the proof of Lemma 2. Furthermore, there are only n choices for $P_{i,\ell}$ and $P_{i,\ell+1}$ leading to case (iii), which is even smaller than the $\hat{\alpha}_0(n) = O(n^{2-\delta})$ number of choices for these indices that lead to case (ii). Thus case (iii) also represents an $o(1)$ contribution. \square

Hence one may replace $S_n(\pi)$ by $PS_n(\pi)$ in (28). Given $\pi \in \mathcal{DNP}P_{[k_1] \cup [k_2]}^m$ and $\mathbf{P} \in PS_n(\pi)$ contributing to this sum, by repeated applications of the pair reduction and (32), we obtain unique reduced $\hat{\pi} \in \mathcal{DNP}P_{[m] \cup [m]}^m$ and $\hat{\mathbf{P}} \in S_n(\hat{\pi})$. (Note that $PS_n(\hat{\pi}) = S_n(\hat{\pi})$, as shown in the proof of Lemma 4.) Furthermore, by (6) we have

$$(35) \quad C_2(a_n(\mathbf{P}_1), a_n(\mathbf{P}_2)) = s^{k-2m} C_2(a_n(\hat{\mathbf{P}}_1), a_n(\hat{\mathbf{P}}_2)),$$

because $a_n(p, q) = \overline{a_n(q, p)}$ so each 2-block $(i, \ell) \sim_\pi (i, \ell')$ contributes a factor

$$\mathbb{E}(|a_n(P_{i,\ell})|^2) = s^2$$

to the covariance. The next lemma assures that the remaining indices in $\hat{\mathbf{P}}$ can vary freely.

Lemma 5. *Given $\pi \in \mathcal{DNP}P_{[k_1] \cup [k_2]}^m$ and $\mathbf{Q} \in S_n(\hat{\pi})$, let*

$$PS_n(\pi; \mathbf{Q}) = \#\{\mathbf{P} \in PS_n(\pi) \mid \hat{\mathbf{P}} = \mathbf{Q}\}.$$

Then

$$(36) \quad \frac{1}{n^{\frac{k}{2}-m}} \#PS_n(\pi; \mathbf{Q}) = 1 + o(1).$$

Proof. With $m > 0$ and $\pi_0 \in \mathcal{DNP}P_m^{[m] \cup [m]}$ fixed, let us prove (36) by induction on $k = k_1 + k_2$ for π with $\hat{\pi} = \pi_0$. The smallest possible value of k is $k = 2m$, for which $\pi = \hat{\pi} = \pi_0$ so (36) is trivial (and holds without the $o(1)$ term).

Thus suppose (36) is known for $k = 2m + 2(j - 1)$ with $j \geq 1$ and consider a partition $\pi \in \mathcal{D}\mathcal{N}\mathcal{P}\mathcal{P}_m^{[k_1] \cup [k_2]}$ with $k_1 + k_2 = 2m + 2j$. Let $(i, \ell) \sim_\pi (i, \ell + 1)$ be a nearest neighbor pair and let $\mathbf{P}' \in PS_n(\pi'; \mathbf{Q})$ be a multi-index for the reduced partition π' with this pair removed. Lifting this multi-index to $[k_1] \cup [k_2]$ by (32) specifies all the indices of \mathbf{P} except for $q_{i,\ell} = p_{i,\ell+1}$. We are free to choose this index as we like, except that we should take $q_{i,\ell} \neq p_{i,\ell}$ and $(p_{i,\ell}, q_{i,\ell})$ cannot fall into any of the equivalence classes already used in \mathbf{P}' . There are, however, only $m + j - 1$ blocks in π' , unless $m = 2$ and there are $m + j - 2$ blocks which only improves things. We conclude that there are at least

$$n - 1 - (m + j - 1)\alpha_3(n) \geq n - 1 - (m + j - 1)\alpha_2(n) = n - O(n^\epsilon)$$

choices for $q_{i,\ell} = p_{i,\ell+1}$. Thus by (C1)

$$(n - O(n^\epsilon)) \times \#PS_n(\pi'; \mathbf{Q}) \leq \#PS_n(\pi; \mathbf{Q}) \leq n \times \#PS_n(\pi'; \mathbf{Q}).$$

Since $n^{-m-j+1} \times \#PS_n(\pi'; \mathbf{Q}) = 1 + o(1)$ by hypothesis, (36) holds for π . \square

Thus we can reduce our considerations to the dihedral partitions in $\mathcal{D}\mathcal{N}\mathcal{P}\mathcal{P}_{[m] \cup [m]}^m$ for $m \geq 1$. As discussed in the paragraph preceding Theorem 3, the map $g \in D_{2m} \mapsto \hat{\pi}_g$ defined for $m \geq 3$ by $(1, \ell) \sim_{\pi_g} (1, g(\ell))$ is a bijection of the dihedral group D_{2m} with $\mathcal{D}\mathcal{N}\mathcal{P}\mathcal{P}_{[m] \cup [m]}^m$. For $m = 1, 2$, it is convenient to define $D_{2m} = \{1_m\}$ to be the one element group and to extend the above map by letting $\hat{\pi}_{1_m}$ be the unique element of $\mathcal{D}\mathcal{N}\mathcal{P}\mathcal{P}_{[m] \cup [m]}^m$ (the trivial partition).

Any consistent multi-index \mathbf{P} compatible with the unique dihedral partition $\hat{\pi}_{1_1} \in \mathcal{D}\mathcal{N}\mathcal{P}\mathcal{P}_{[1] \cup [1]}^1$ is diagonal,

$$\mathbf{P} = (P_{1,1}, P_{2,1}), \text{ with } P_{1,1} = (p, p), P_{2,1} = (q, q) \text{ and } (p, p) \sim_{\pi_{1_1}} (q, q).$$

In contrast, for $m \geq 2$ we shall now show that diagonal indices give negligible contribution. That is, given $g \in D_{2m}$ with $m \geq 2$, we let $S_n^{\text{OD}}(\hat{\pi}_g)$ denote the set of $\mathbf{P} = ((p_{i,\ell}, q_{i,\ell})) \in S_n(\pi_g)$ with $p_{i,\ell} \neq q_{i,\ell}$ for $i = 1, 2$ and $\ell = 1, \dots, m$. Then

Lemma 6. *Let $g \in D_{2m}$, $m \geq 2$, then*

$$\frac{1}{n^m} \#S_n(\hat{\pi}_g) = \frac{1}{n^m} \#S_n^{\text{OD}}(\hat{\pi}_g) + o(1).$$

Proof. Let us count the number of multi-indices \mathbf{P} with a diagonal index $P_{i,\ell} = (p, p)$ at given position i, ℓ . There are n choices for p , and thus no more than n^{m-1} choices of the indices on circle i , since proceeding cyclically to $P_{i,\ell+1}, P_{i,\ell+2}$, etc., the last index $P_{i,\ell-1}$ is completely specified by consistency. Since every point is a connector, there remain only $\alpha_2(n)^m$ choices for the indices on the other circle, giving a total of $n^{m-1} \times \alpha_2(n)^m$ choices for \mathbf{P} with $P_{i,\ell} = (p, p)$.

Now every $\mathbf{P} \in S_n(\hat{\pi}_g) \setminus S_n^{\text{OD}}(\hat{\pi}_g)$ has some diagonal index, with $2m$ possible positions for this index. We conclude that

$$\#S_n(\hat{\pi}_g) \setminus S_n^{\text{OD}}(\hat{\pi}) \leq 2mn^{m-1}\alpha_2(n)^m = O(n^{m-\delta}),$$

and the lemma follows. \square

Putting all of these results together — using (35) in (28), and then applying Lemmas 4 and 6 to replace the sum over $\mathbf{P} \in S_n(\pi)$ by a sum over $\hat{\mathbf{P}} \in S_n^{\text{OD}}(\hat{\pi})$ — we find that

$$\begin{aligned} C_2(X_n^{k_1}, X_n^{k_2}) \\ = \sum_{m=1}^{\min\{k_1, k_2\}} \frac{s^{k-2m} A_{k_1, k_2}^m}{n^m} \sum_{g \in D_{2m}} \sum_{\hat{\mathbf{P}} \in S_n^{\text{OD}}(\hat{\pi}_g)} C_2(a_n(\hat{\mathbf{P}}_1), a_n(\hat{\mathbf{P}}_2)) + o(1), \end{aligned}$$

where

$$A_{k_1, k_2}^m = \#\{\pi \in \mathcal{D}\mathcal{N}\mathcal{P}\mathcal{P}_{[k_1] \cup [k_2]}^m \mid \hat{\pi} = \hat{\pi}_g\},$$

which we shall see does not depend on $g \in D_{2m}$, and for $m = 1$ we let $S_n^{\text{OD}}(\hat{\pi}_{1_1}) = S_n(\hat{\pi}_{1_1})$.

To derive an expression for A_{k_1, k_2}^m , we decompose $\pi \in \mathcal{D}\mathcal{N}\mathcal{P}\mathcal{P}_{[k_1] \cup [k_2]}^m$ into two partitions π_1, π_2 of $[k_1], [k_2]$ respectively by cutting all links between connectors, following [KMS]. That is,

$$(37) \quad \ell \sim_{\pi_i} \ell' \Leftrightarrow (i, \ell) \sim_\pi (i, \ell') \text{ and } (i, \ell), (i, \ell') \text{ are not connectors.}$$

(Note that for $\pi \in \mathcal{D}\mathcal{N}\mathcal{P}\mathcal{P}_{[k_1] \cup [k_2]}^2$ we decompose the 4-block into 4 singletons). The resulting partitions π_i , $i = 1, 2$, are called *non-crossing half pair partitions*. Here a *half pair partition* of $[k]$ is a partition consisting of 2-blocks (pairs) and 1-blocks, called *open connectors*, and in analogy with the case of pair partitions, we call a half pair partition π *non-crossing* if for every pair $\ell_1 \sim_\pi \ell_2$ and any points $\ell'_1 \in]\ell_1, \ell_2[, \ell'_2 \in]\ell_2, \ell_1[$ we have $\ell'_1 \not\sim_\pi \ell'_2$ and at most one of ℓ'_1, ℓ'_2 is an open connector. We denote the set of non-crossing half pair partitions with exactly m open connectors by $\mathcal{N}\mathcal{H}\mathcal{P}\mathcal{P}_{[k]}^m$.

Thus, given $\pi \in \mathcal{D}\mathcal{N}\mathcal{P}\mathcal{P}_{[k_1] \cup [k_2]}^m$ we have maps: $\pi \mapsto \hat{\pi} \in \mathcal{D}\mathcal{N}\mathcal{P}\mathcal{P}_{[m] \cup [m]}^m$ and $\pi \mapsto \pi_i \in \mathcal{N}\mathcal{H}\mathcal{P}\mathcal{P}_{[k_i]}^m$, $i = 1, 2$. Conversely, given a triple of partitions $\pi_0 \in \mathcal{D}\mathcal{N}\mathcal{P}\mathcal{P}_{[m] \cup [m]}^m$ and $\pi_i \in \mathcal{N}\mathcal{H}\mathcal{P}\mathcal{P}_{[k_i]}^m$, $i = 1, 2$, we can combine them into a partition

$$(38) \quad \pi_1 \circ_{\pi_0} \pi_2 \in \mathcal{D}\mathcal{N}\mathcal{P}\mathcal{P}_{[k_1] \cup [k_2]}^m$$

by attaching the open connectors on each circle according to π_0 : if $\ell_{i,1} < \ell_{i,2} < \dots < \ell_{i,m} \in [k_i]$ are the open connectors under π_i , then

$$(i, \ell) \sim_{\pi_1 \circ_{\pi_0} \pi_2} (i', \ell') \Leftrightarrow \begin{cases} i = i' \text{ and, } (i, \ell) \sim_{\pi_i} (i, \ell'), \text{ or} \\ \ell = \ell_{i,j} \text{ and } \ell' = \ell_{i',j'} \text{ with } (i, j) \sim_{\pi_0} (i', j'). \end{cases}$$

If we decompose $\pi \in \mathcal{D}\mathcal{N}\mathcal{P}\mathcal{P}_{[k_1] \cup [k_2]}^m$ and then recombine the corresponding triple we obtain π again,

$$\pi \mapsto (\hat{\pi}, \pi_1, \pi_2) \mapsto \pi_1 \circ_{\hat{\pi}} \pi_2 = \pi.$$

In particular we see that

$$A_{k_1, k_2}^m = \#\mathcal{N}\mathcal{H}\mathcal{P}\mathcal{P}_{[k_1]}^m \times \#\mathcal{N}\mathcal{H}\mathcal{P}\mathcal{P}_{[k_2]}^m.$$

Thus, let us set $t_{k,m} = \#\mathcal{NHP}\mathcal{P}_{[k]}^m$. The results of this section are then summarized by

$$(39) \quad C_2(X_n^{k_1}, X_n^{k_2}) \\ = \sum_{m=1}^{\min\{k_1, k_2\}} \frac{t_{k_1, m} t_{k_2, m}}{n^m} s^{k_1+k_2-2m} \sum_{\substack{g \in D_{2m} \\ \mathbf{P} \in S_n^{\text{OD}}(\hat{\pi}_g)}} C_2(a_n(\mathbf{P}_1), a_n(\mathbf{P}_2)) + o(1),$$

with $s^2 = \mathbb{E}(|a|^2)^{\frac{1}{2}}$. We note that, for $\mathbf{P} \in S_n^{\text{OD}}(\hat{\pi}_g)$ with $g \in D_{2m}$ for $m \neq 2$,

$$(40) \quad C_2(a_n(\mathbf{P}_1), a_n(\mathbf{P}_2)) = \prod_{\ell=1}^m \mathbb{E}(a_n(P_{1,\ell}), a_n(P_{1,g(\ell)})),$$

by (30) and the independence of elements from distinct classes. For $m = 2$, with $\mathbf{P} \in S_n^{\text{OD}}(\hat{\pi}_{1_2})$,

$$(41) \quad \begin{aligned} C_2(a_n(\mathbf{P}_1), a_n(\mathbf{P}_2)) &= C_2(|a_n(P_{1,1})|^2, |a_n(P_{2,1})|^2) \\ &= \mathbb{E}(|a_n(P_{1,1})|^2 |a_n(P_{2,1})|^2) - \mathbb{E}(|a_n(P_{1,1})|^2) \mathbb{E}(|a_n(P_{2,1})|^2), \end{aligned}$$

since $a_n(P_{i,1}) = \overline{a_n(P_{i,2})}$ in this case.

5. NON-CROSSING HALF PAIR PARTITIONS AND CHEBYSHEV POLYNOMIALS

As just became apparent, we have to control the number of non-crossing half pair partitions. This can be done by a “low-tech” version of the arguments in [KMS]. As our simplified derivation has not appeared elsewhere to our knowledge, we include the details for the sake of completeness of the present work. This allows us to complete the proof of Theorem 3.

Before proceeding, it is convenient extend the definition of the non-crossing half pair partitions to include those with “no connector,” $\mathcal{NHP}\mathcal{P}_{[k]}^0$, (i.e., $m = 0$). It turns out that the useful object here is *not* simply the set of non-crossing pair partitions of $[k]$, but is instead the set of such partitions furnished with a marked point:

$$(42) \quad \mathcal{NHP}\mathcal{P}_{[k]}^0 = \{(\pi, \mu) \in \mathcal{NPP}_{[k]} \times [k] \mid \mu \neq k \Rightarrow \mu + 1 \sim_\pi \ell \text{ with } \ell \leq \mu\}.$$

Here $\mathcal{NPP}_{[k]}$ is the set of pair partitions of $[k]$ which are non-crossing in the sense that for any pair $m_1 \sim_\pi m_2$ if $m'_1 \in]m_1, m_2[$ and $m'_2 \in]m_2, m_1[$, then $m'_1 \not\sim_\pi m'_2$.

The combinatorics of non-crossing half pair partitions is controlled by the coefficients $T_{m,k}$ of the Chebyshev polynomials of the first kind as defined in (18). Recall that $T = (T_{m,k})$ is an infinite lower triangular matrix with ones on the diagonal. Thus T has a unique lower triangular inverse. The main result of this section is that $T^{-1} = (t_{k,m})$, with $t_{k,m}$ the coefficients that appear in (39):

Theorem 4. *For $m \geq 0$, let us set $t_{k,m} = \#\mathcal{NHP}\mathcal{P}_{[k]}^m$ for $k \geq 1$ and $t_{0,m} = \delta_{m,0}$. Then*

$$\sum_k T_{m,k} t_{k,m'} = \delta_{m,m'}.$$

In other words the infinite lower triangular matrices $T = (T_{m,k})_{m,k \geq 0}$ and $t = (t_{k,m})_{k,m \geq 0}$ are inverses of each other.

This theorem allows to complete the

Proof of Theorem 3: Using multi-linearity of the cumulants and the power expansion (18) of the Chebyshev polynomials, Theorem 3 follows directly from (39), (40) and (41) and Theorem 4. \square

In order to prove Theorem 4 we use the three term recurrence relation satisfied by the monic Chebyshev polynomials:

$$(43) \quad xT_m(x) = T_{m+1}(x) + (1 + \delta_{m,1})T_{m-1}(x), \quad m \geq 0$$

with $T_{-1} \equiv 0$ by convention.

Proof of Theorem 4. Expressed in terms of the coefficients, the three-term recurrence relation (43) reads

$$(44) \quad T_{m,k-1} = T_{m+1,k} + (1 + \delta_{m,1})T_{m-1,k},$$

with boundary conditions $T_{-1,k} = T_{m,-1} = 0$. We may write (44) in operator notation as

$$(45) \quad TS = ST + S^*(\mathbf{1} + P_0)T,$$

where we consider the infinite lower triangular matrix T as an operator on sequences $(\phi(0), \phi(1), \dots) \in \ell^2(\mathbb{N})$, i.e., $T\phi(m) = \sum_{k=0}^m T_{m,k}\phi(k)$. Here S is the backwards shift, with S^* its adjoint,

$$S\phi(k) = \phi(k+1), \quad S^*\phi(k) = (1 - \delta_{k,0})\phi(k-1),$$

and P_0 is the projection onto $\delta_{k,0}$:

$$P_0\phi(k) = \delta_{k,0}\phi(0).$$

As T is invertible, (45) implies

$$ST^{-1} = T^{-1}S + T^{-1}S^*(\mathbf{1} + P_0).$$

Expressed in terms of matrix elements this reads

$$(46) \quad (T^{-1})_{k+1,m} = (T^{-1})_{k,m-1} + (1 + \delta_{m,0})(T^{-1})_{k,m+1},$$

with the boundary condition $(T^{-1})_{k,-1} = 0$. Since $t_{0,0} = (T^{-1})_{0,0} = 1$, to complete the proof we need only to show that the numbers $t_{k,m}$ satisfy the same recurrence relation as $(T^{-1})_{k,m}$, that is

$$(47) \quad \#\mathcal{NHP}^0_{[k+1]} = 2 \times \#\mathcal{NHP}^1_{[k]}, \quad \text{and}$$

$$(48) \quad \#\mathcal{NHP}^m_{[k+1]} = \#\mathcal{NHP}^{m+1}_{[k]} + \#\mathcal{NHP}^{m-1}_{[k]}, \quad m \geq 1.$$

For $m \geq 1$, it is sufficient to present a bijection,

$$\mathcal{Z} : \mathcal{NHP}^m_{[k+1]} \rightarrow \mathcal{NHP}^{m+1}_{[k]} \cup \mathcal{NHP}^{m-1}_{[k]}.$$

One such map can be constructed as follows:

(I) Let $\ell_{\max}(\pi)$ be the largest connector of π and define $\mu(\pi)$ to be the largest element of the set

$$\left\{ \ell \in [\ell_{\max}(\pi), k+1] \mid \text{every point in }]\ell_{\max}(\pi), \ell] \right. \\ \left. \text{is paired by } \pi \text{ to a point in }]\ell_{\max}(\pi), \ell] \right\}.$$

(The set is non-empty and $\mu(\pi)$ exists because it contains $\ell_{\max}(\pi)$, for which the condition is vacuous.)

(II) Define $\mathcal{Z}\pi$ to be the partition of $[k]$ constructed by removing $\mu(\pi)$, and relabeling $\ell \mapsto \ell - 1$ for $\ell > \mu(\pi)$. That is,

$$(49) \quad \ell \sim_{\mathcal{Z}\pi} \ell' \Leftrightarrow r_\pi(\ell) \sim_\pi r_\pi(\ell'),$$

with

$$r_\pi(\ell) = \begin{cases} \ell & \ell < \mu(\pi), \\ \ell + 1 & \ell \geq \mu(\pi) \end{cases}, \quad r_\pi : [k] \rightarrow [k+1].$$

If $m = 1$ and $\mu(\pi) = \ell_{\max}(\pi)$ the resulting partition has no open connector. To obtain an element of $\mathcal{NHP}\mathcal{P}_{[k]}^0$ we mark the point $\mu(\pi) - 1$.

To show that \mathcal{Z} is a bijection, it suffices to exhibit its inverse. The key fact to note here is that

$$\mu(\mathcal{Z}\pi) = \mu(\pi) - 1,$$

where for $\pi \in \mathcal{NHP}\mathcal{P}_{[k]}^0$ we let $\mu(\pi)$ denote the marked point of π . Thus for $\pi \in \mathcal{NHP}\mathcal{P}_{[k]}^{m \pm 1}$, the image $\mathcal{Z}^{-1}\pi$ is obtained by inserting a point to the right of $\mu(\pi)$. The new point is an open connector or is connected to the largest open connector of π depending on whether π has $m - 1$ or $m + 1$ connectors, respectively. That is, the two branches of the inverse $\mathcal{Z}_\pm^{-1} : \mathcal{NHP}\mathcal{P}_{[k]}^{m \pm 1} \rightarrow \mathcal{NHP}\mathcal{P}_{[k+1]}^m$ satisfy

$$(50) \quad \ell \sim_{\mathcal{Z}_-^{-1}\pi} \ell' \Leftrightarrow \begin{cases} \ell = \ell' = \mu(\pi) + 1, \text{ or} \\ \ell, \ell' \in [k] \setminus \{\mu(\pi) + 1\} \text{ and } f_\pi(\ell) \sim_\pi f_\pi(\ell'), \end{cases}$$

$$(51) \quad \ell \sim_{\mathcal{Z}_+^{-1}\pi} \ell' \Leftrightarrow f_\pi(\ell) \sim_\pi f_\pi(\ell'),$$

with

$$f_\pi(\ell) = \begin{cases} \ell & \ell \leq \mu(\pi) \\ \ell_{\max}(\pi) & \ell = \mu(\pi) + 1, \\ \ell - 1 & \ell > \mu(\pi) \end{cases}, \quad f_\pi : [k+1] \rightarrow [k].$$

For $m = 0$ a separate argument is needed. The map \mathcal{Z} , that is removal of $\mu(\pi)$, extends to this case and (49) defines a map $\mathcal{Z} : \mathcal{NHP}\mathcal{P}_{[k+1]}^0 \rightarrow \mathcal{NHP}\mathcal{P}_{[k]}^1$. (Recall that $\mu(\pi)$ is the marked point of π .) However, \mathcal{Z} is *not* a bijection on $\mathcal{NHP}\mathcal{P}_{[k+1]}^0$. We will show that it is a double cover, from which (47) follows. To see this, let us define a map $\tilde{\mathcal{Z}} : \mathcal{NHP}\mathcal{P}_{[k+1]}^0 \rightarrow \mathcal{NHP}\mathcal{P}_{[k]}^1 \times \{-1, +1\}$ with

$$\tilde{\mathcal{Z}}\pi = (\mathcal{Z}\pi, \sigma), \quad \sigma = \begin{cases} 1 & \text{if } \mu(\pi) \sim_\pi \ell \text{ with } \ell < \mu(\pi), \\ -1 & \text{otherwise.} \end{cases}$$

We claim that $\tilde{\mathcal{Z}}$ is a bijection (so \mathcal{Z} is a double cover). First note the definition (51) of $\mathcal{Z}_+^{-1}\pi$, *i.e.*, insertion of a point to the right of $\mu(\pi)$ and paired with $\ell_{\max}(\pi)$, gives a map $\mathcal{Z}_+^{-1} : \mathcal{NHP}\mathcal{P}_{[k]}^1 \rightarrow \mathcal{NHP}\mathcal{P}_{[k+1]}^0$. (We mark the inserted point $\mu(\pi) + 1$ of $\mathcal{Z}_+^{-1}\pi$.) Furthermore $\tilde{\mathcal{Z}}\mathcal{Z}_+^{-1}\pi = (\pi, +1)$, so \mathcal{Z}_+^{-1} is one branch of the inverse $\tilde{\mathcal{Z}}^{-1}$. To construct the other branch, note that for $\pi \in \mathcal{NHP}\mathcal{P}_{[k+1]}^0$ the following dichotomy holds: either $\mu(\pi) \sim_\pi \ell$ with $\ell < \mu(\pi)$ or $\mu(\pi) \sim_\pi \mu(\pi) + 1$. Thus $\tilde{\mathcal{Z}}^{-1}(\pi, -1)$ is the partition obtained by inserting a marked point paired with and immediately to the *left* of the connector. \square

6. EVALUATING THE COVARIANCE

The aim of this section is first to prove Theorem 1 and second to verify the limiting covariance (4) for the Wigner ensemble with complex matrix entries. The main point is to show by example how Theorem 3 can be used to calculate the covariance for specific random matrix ensembles.

Proof of Theorem 1. By Corollary 3, it suffices to show

$$\begin{aligned} [m=1] \quad & \frac{1}{n} \# \{(p, q) \mid (p, p) \sim_n (q, q)\} \rightarrow T, \\ [m=2] \quad & \frac{1}{n^2} \# \{(p, q, p', q') \mid p \neq q, p' \neq q', \text{ and } (p, q) \sim_n (p', q')\} \rightarrow 2T, \\ [m \geq 3] \quad & \frac{1}{n^m} \# S_n^{\text{OD}}(\hat{\pi}_g) \rightarrow T, \text{ for } g \in D_{2m}. \end{aligned}$$

For $m = 1$, we have

$$\{(p, q) \mid (p, p) \sim_n (q, q)\} = \{(p, \phi_n^t(p)) \mid t = 0, \dots, T-1\}.$$

Thus

$$\# \{(p, q) \mid (p, p) \sim_n (q, q)\} \leq Tn,$$

and

$$\begin{aligned} \# \{(p, q) \mid (p, p) \sim_n (q, q)\} &\geq \# \{p \mid \phi_n^t(p) \neq p \text{ for } t = 1, \dots, T-1\} \\ &= T(n - o(n)) \end{aligned}$$

since $\# \{p \mid \phi_n^t(p) = p \text{ for some } t = 1, \dots, T-1\} = o(n)$.

For $m = 2$, we have

$$\begin{aligned} & \{(p, q, p', q') \mid p \neq q, p' \neq q', \text{ and } (p, q) \sim_n (p', q')\} \\ &= \{(p, q, \phi_n^t(p), \phi_n^t(q)) \mid p \neq q, \text{ and } t = 0, \dots, T-1\} \\ &\quad \cup \{(p, q, \phi_n^t(q), \phi_n^t(p)) \mid p \neq q, \text{ and } t = 0, \dots, T-1\}. \end{aligned}$$

(Since ϕ_n is a bijection $p \neq q \Leftrightarrow \phi_n^t(p) \neq \phi_n^t(q)$.) Thus

$$\# \{(p, q, p', q') \mid p \neq q, p' \neq q', \text{ and } (p, q) \sim_n (p', q')\} \leq 2Tn(n - o(n)).$$

To obtain a lower bound, choose p arbitrarily and then q such that $q \neq \phi_n^t(p)$ for $t = 0, \dots, T-1$ and $q \neq \phi_n^t(q)$ for $t = 1, \dots, T-1$. Given such a pair we have $(p, q) \neq (\phi_n^t(p), \phi_n^t(q))$ and $(q, p) \neq (\phi_n^t(q), \phi_n^t(p))$, for $t = 1, \dots, T-1$, and further $(p, q) \neq (\phi_n^t(q), \phi_n^t(p))$, for $t = 0, \dots, T-1$. It follows that the $2T$ indices $(p, q, \phi_n^t(p), \phi_n^t(q))$, $(p, q, \phi_n^t(q), \phi_n^t(p))$ with $t = 0, \dots, T-1$ are distinct elements of the set to be counted. There are n choices of p followed by at least $n - T - o(n)$ choices for q , giving

$$\# \{(p, q, p', q') \mid p \neq q, p' \neq q', \text{ and } (p, q) \sim_n (p', q')\} \geq 2Tn(n - o(n)).$$

Finally, for $m = 3$, it suffices to let $g = 1_m$ be the identity in D_{2m} , so $(1, \ell) \sim_{\hat{\pi}_{1_m}} (2, \ell)$ for $\ell = 1, \dots, m$. It is useful to introduce the compact notation

$$P_{1,\ell} = (p_\ell, p_{\ell+1}), \quad P_{2,\ell} = (q_\ell, q_{\ell+1}), \quad \ell = 1, \dots, m$$

for the elements of a consistent multi-index, with $\ell + 1$ computed modulo m (so $m + 1 = 1$). First consider the set $PS_n^{\text{OD}}(\hat{\pi}_{1_m})$ of multi-indices with $q_\ell = \phi_n^t(p_\ell)$ for some fixed $t = 0, \dots, T-1$. Clearly $\# PS_n^{\text{OD}}(\hat{\pi}_{1_m}) \leq Tn^m$. To obtain a lower bound, restrict the indices p_ℓ to be in the set $\{p \mid \phi_n^t(p) \neq p \text{ for } t = 1, \dots, T-1\}$ and further demand that p_ℓ avoid the orbit under ϕ_n of all $p_{\ell'}$ with $\ell' < \ell$. In

this way we guarantee that each choice of $t = 0, 1, \dots, T$ gives a distinct $\hat{\pi}_{1_m}$ -compatible multi-index. Thus $\#PS_n^{\text{OD}}(\hat{\pi}_{1_m}) \geq T(n - o(n))^m$ and we conclude that $n^{-m} PS_n^{\text{OD}}(\hat{\pi}_{1_m}) \rightarrow 1$.

Thus, we must show that

$$(52) \quad \frac{1}{n^m} \#S_n^{\text{OD}}(\hat{\pi}_{1_m}) \setminus PS_n^{\text{OD}}(\hat{\pi}_{1_m}) = 0.$$

For this purpose it is useful to consider two classes of multi-indices \mathbf{P} which are easily seen to cover $S_n^{\text{OD}}(\hat{\pi}_{1_m}) \setminus PS_n^{\text{OD}}(\hat{\pi}_{1_m})$:

- (I) $(q_{\ell-1}, q_\ell) = (\phi_n^t(p_{\ell-1}), \phi_n^t(p_\ell))$ and $(q_\ell, q_{\ell+1}) = (\phi_n^{t'}(p_\ell), \phi_n^{t'}(p_{\ell+1}))$ for some ℓ and $t \neq t'$, or
- (II) $(q_{\ell-1}, q_\ell) = (\phi_n^t(p_\ell), \phi_n^t(p_{\ell-1}))$ for some ℓ and t .

In case (I), we have $\phi_n^t(p_\ell) = \phi_n^{t'}(p_\ell)$. Thus $p_\ell = \phi_n^{t-t'}(p_\ell)$ with $t - t'$ not a multiple of T and there are only $o(n)$ possible values for p_ℓ . Given this index, we label the other indices on circle one arbitrarily, using the connectors to label circle two to find that there are no more than

$$m \times o(n) \times n^{m-1} \times \alpha_2(n)^m = o(n^m)$$

such multi-indices, since $\alpha_2(n) = 2T$ is bounded. (The factor m is the number of possible values of ℓ .) Thus the contribution from this case is $o(1)$. In case (II), one gets from the pair $(1, \ell) \sim_{\pi_g} (2, \ell)$ the condition $(p_\ell, p_{\ell+1}) \sim_n (\phi_n^t(p_{\ell-1}), q_{\ell+1})$. We conclude that either p_ℓ or $p_{\ell+1}$ is in the orbit of $p_{\ell-1}$ under ϕ_t . Therefore, there are no more than $T \times n^2$ choices for the indices $p_{\ell-1}, p_\ell, p_{\ell+1}$. Thus there are no more than

$$m \times Tn^2 \times n^{m-3} \times \alpha_2(n)^m = O(n^{m-1})$$

multi-indices in case (II). Since this contribution is also $o(1)$, eq. (52) holds and the corollary follows. \square

When the entries of the matrix are complex, the covariance is quite a bit more complicated. This is already the case for Wigner matrices with $T = 1$, for which (4) holds. To indicate the differences between the complex and real case let us sketch the proof of (4).

Proof of equation (4). Following the proof of Corollary 3, we find that for the identity $1_m \in D_{2m}$ the dominant contribution comes from multi-indices in which the indices on the two circles are equal: $P_{1,\ell} = P_{2,\ell}$. Thus the dominant contribution for $g \in D_{2m}$ is the image of this set under the action of g defined in the proof of Theorem 1 (see (19)). We now consider separately the contribution from reflections and rotations.

For a reflection $g \in D_{2m}$, the order of indices on the second circle is reversed in (19). Thus we find that every matrix element is paired with its conjugate, resulting in a factor $\mathbb{E}(|a|^2)^m$ from the expectation. Since there are m reflections, this gives the first term in the formula (4) for the variance.

The m rotations also give identical contributions, since one easily checks that $\mathbb{E}(a_n(g \cdot \mathbf{P})) = \mathbb{E}(a_n(\mathbf{P}))$ for g a rotation. However $\mathbb{E}(a_n(g \cdot \mathbf{P})) = \mathbb{E}(a^2)^k \mathbb{E}(\bar{a}^2)^{m-k}$

with k depending on the value of $\mathbf{P} \in S_n^{\text{OD}}(\hat{\pi}_g)$. The asymptotics of the appearance of these terms is governed by the probability distribution ρ_m given by

$$\rho_m(k) = \text{Vol}_m \{x \in [0, 1]^m \mid k \text{ rises in } x_1, x_2, \dots, x_m, x_1\},$$

where $x = (x_1, \dots, x_m)$ and a *rise* means that $x_l > x_{l-1}$. This combinatorial integral can be calculated as in [Foa]. Let the set to be integrated be called $V_{m,k}$. Then $x \in V_{m,k}$ if and only if there exists a permutation $\sigma \in S_m$ with k cyclic rises such that $x_{\sigma^{-1}(1)} < \dots < x_{\sigma^{-1}(m)}$. Hence $\rho_m(k)$ is equal to $1/m!$ times the number of permutations with k cyclic rises. The latter is m times the number of permutations with k cyclic rises and the property $\sigma(1) = 1$. The number of permutations $\sigma \in S_m$ with $\sigma(1) = 1$ and k rises is known to be given by the Eulerian numbers $A_{m-1,k}$ where

$$A_{m,k} = \sum_{j=0}^k (-1)^j (k-j)^m \frac{(m+1)!}{j!(m+1-j)!}.$$

This completes the proof. \square

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